Text Problems:

• Section 11.2: 19
• Section 11.3: 14

Other Problems

P1 (Eigenfunction method; inhomogeneous BCs). Consider the following IBVP for the heat equation with inhomogeneous Neumann BCs:

\[ u_t = u_{xx}, \quad x \in [0,1], \quad t > 0, \]

\[ u_x(0,t) = 0, \quad u_x(1,t) = -t, \quad t > 0, \]

\[ u(x,0) = 1. \]

a) Find \( u(x,t) \) using the eigenfunction method directly. Evaluate all coefficients explicitly. \textit{Note: be careful with the zero eigenvalue!}

b) Describe the behavior of the solution as \( t \to \infty \).

c) Find \( u(x,t) \) by using the trick from class to move boundary conditions to a source (i.e. find \( w(x,t) \) that satisfies the boundary conditions and solve for \( v = u - w \)). \textit{Hint: Try a line first; if that doesn’t work try a quadratic.}

You may leave your answer in terms of \( a_n \) where

\[ a_n = \int_0^1 x^2 \cos n\pi x \, dx = \frac{2\cos \pi n}{\pi^2 n^2}, \quad n \geq 1. \]
P2 (Wave equation; resonance from a BC). Consider a string of length \( \pi \) that is fixed at one end. It is driven by oscillating the other end at a frequency \( \omega \). The displacement \( u \) of the string is governed by the IBVP

\[
\begin{align*}
  u_{tt} &= u_{xx}, & x &\in [0, \pi], & t &> 0, \\
  u(0, t) &= \pi \sin \omega t, & u(\pi, t) &= 0, & t &> 0, \\
  u(x, 0) &= 0, & u_t(x, 0) &= 0.
\end{align*}
\]

a) When do you expect solutions to stay bounded or grow over time?

b) Find \( u(x, t) \) when the frequency \( \omega = N \) (a positive integer). Define all coefficients explicitly. Does this agree with your prediction in part (a)?

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(optional) P3: Start-up flow in a circular pipe. Note: the context here is not needed to solve the problem, so you do not need to pay much attention to it except to know which variables are just constants. Pay close attention to the things you can and cannot obtain explicitly (this is a useful skill since it comes up often in practical problems).

Suppose a fluid is sent through a circular pipe of radius 1 by applying a constant pressure gradient\(^1\). At the start, the fluid is at rest; eventually, the fluid will reach a steady flow (that does not change in time). In this problem we will determine the behavior of this flow before it reaches its steady state\(^2\).

Let us assume that the fluid velocity \( u(r, t) \) down the pipe depends on \( r \) and \( t \) (so the profile is the same at each cross-section and the velocity always parallel to the pipe).

The governing equation (after some rescaling) can be shown to be

\[
  u_t = G + u_{rr} + \frac{1}{r} u_r, \quad r \in [0, 1), \quad t > 0,
\]

with boundary condition

\[
  u(1, t) = 0, \quad t > 0
\]

and initial condition

\[
  u(r, 0) = 0.
\]

Here \( G > 0 \) is a constant that describes the pressure gradient.

a) Find the steady state \( \bar{u}(r) \). \textit{Hint: the ODE is a first-order equation for} \( \bar{u}' \).

b) Let \( v = u - \bar{u} \) be the ‘transient’ part of the velocity (which we expect to go to zero as \( t \to \infty \)). Write down the initial boundary value problem (PDE, initial and boundary conditions) to be solved for \( v \).

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\(^1\)Problem adapted from Langlois & Deville, \textit{Slow Viscous Flow}.

\(^2\)This steady state is called a \textit{Poiseuille flow}. 
c) Our goal is to obtain the transient part \( v(r,t) \) in the form

\[
v(r,t) = \sum_{n=1}^{\infty} g_n(t) \phi_n(r).
\]

Assume that the \( \lambda_n \)'s are positive and set \( \mu_n = \sqrt{\lambda_n} \). Show (by whatever method) that the \( \phi_n \)'s and \( \mu_n \)'s are the solutions to

\[
\phi'' + \frac{1}{r} \phi' + \mu^2 \phi = 0.
\]

(1)

What are the two boundary conditions for this problem?

d) For this part, you should use the facts about Bessel functions listed at the end of the problem. Let \( z = \mu r \) and show that this transformation turns the ODE into Bessel’s equation of order zero.

Use this to show that the \( \mu_n \)'s are the positive zeros of \( J_0 \) and \( \phi_n(r) = J_0(\mu_n r) \) for \( n = 1, 2, \ldots \).

e) Write (1) in ‘regular’ form

\[
L[\phi] = \lambda w(r) \phi.
\]

Hint: Multiply by \( r \). We know that this operator with the given boundary conditions is self-adjoint (see the Example in 2.3.2 of Week 14). Write down explicitly (i.e. as an integral) the orthogonality relation for the eigenfunctions.

f) Find the transient velocity \( v(r,t) \) in the form given in (c). Your solution should be in terms of \( \lambda_n, J_0 \) and the two quantities

\[
\alpha_n = \int_0^1 r (1 - r^2) J_0(\mu_n r) \, dr, \quad \beta_n = \int_0^1 r (J_0(\mu_n r))^2 \, dr
\]

which are nasty integrals (computing them is not recommended). Verify (formally) that the total velocity \( u(r,t) \) approaches the steady state \( \bar{u} \) as \( t \to \infty \).

Note: For this problem, you will need some facts about Bessel functions.

Bessel’s equation of order zero is

\[
y'' + \frac{1}{x} y' + y = 0.
\]

which has the general solution

\[
y = c_1 J_0(x) + c_2 Y_0(x)
\]

where \( J_0, Y_0 \) are the Bessel functions of the first/second kind. They have the property \(^3\) that

\[
J_0(x) \sim \text{const.}, \quad Y_0(x) \sim C \ln(x) \quad \text{as } x \to 0.
\]

The function \( J_0(x) \) is finite for all \( x \) and has an infinite sequence of positive zeros.

\(^3\)We derived this result in studying Frobenius series!