Math 353, Fall 2017
Study guide for the final

December 12, 2017

Last updated 12/4/17. If something is unclear or incorrect, please let me know so I can update the documents.

Updates

• 12/4: De-emphasized regular singular points (some points in red removed entirely); increased priority of ‘separable equations’ (now blue). Added a note on variation of parameters (the formula is on the formula sheet; just know how to use it).

• 12/12: Removed some regular singular point topics.

Study materials

There are several good sources of practice problems, review and reading:

Reading: Aside from my lecture notes, ODE and PDE Notes I and II (up through Chapter 2 of part II) is a terse source of notes (good for studying core concepts and getting an overview). Much of the material is standard, so other sources (e.g. lecture notes from other classes/universities etc.) could also be helpful.

The book is sometimes a reasonable but not complete resource (except for PDEs in Chapter 11).

Practice problems: On Sakai. Contains a list of suggested book problems (the ones I think would be helpful), including some highlighted problems. There are a few practice problems relating to PDEs

AHP: Almost all of them are good for reviewing/practicing with concepts.

Old Midterms: On Sakai.

For PDEs: Homeworks 9 and 10. Chapter 10.5-10.7 has a few problems, but Chapter 11 is of limited use (most of the problems are not PDEs), particularly for the more advanced material.
For basic ODEs: If you want some ‘random’ first order ODEs to solve just to practice solving them, Section 2.9 of the book has a list.
Some general advice

- Shorthand/notation can save some writing, but be clear. If you use $\phi_n$ for an eigenfunction, then make sure that you clearly state what $\phi_n$ is equal to. If using shorthand notation like $\langle f, g \rangle$ you should define it (e.g. ‘where $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$’). If your final answer is something like
  \[ \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \phi_n(x) \]
  then there should be formulas for $\lambda_n, \phi_n$ and $a_n$ on the page that are easy to find. Of course if there are some arbitrary functions in the problem, these answers will be in terms of those quantities.

- If you are stuck on a computation, you can set it equal to a variable and move on to maximize partial credit. As an example, suppose you are solving
  \[ y'' + 2y' + 2y = \delta(t - 1), \quad y(0) = y'(0) = 0 \]
  and need to inverse transform to get
  \[ y(t) = \mathcal{L}^{-1}\left[ \frac{e^{-s}}{s^2 + 2s + 2} \right] \]
  but you forget how to compute $\mathcal{L}^{-1}[1/(s^2 + 2s + 2)]$. Then you should define
  \[ g(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^2 + 2s + 2} \right] \]
  and then obtain the answer
  \[ y(t) = g(t - 1)u_1(t). \]
  (This gives you partial credit for using the step function rule).

- Knowing the correct form of the solution helps, but for partial credit it needs to be correct! Writing down the Fourier series in general will not do you much good.

- Avoid doing excessive work. If you are allowed to, use the Laplace transform rules when it helps; use the integration tricks for even odd functions, using integration rules from a table if supplied etc.

  Pay attention to what is being asked, and what is not - if asked only to find a steady state, you may be able to do so without obtaining the full solution to the PDE. If asked to show how a Frobenius series solution looks to leading order, you do not have to find the whole series.

- Try to pick the most efficient method to solve the problem. Subtracting out a steady state is the easier way to solve inhomogeneous problems when it works (similarly, separation of variables is the easiest way to solve homogeneous problems). Undetermined coefficients is easier than variation of parameters when the ‘guess’ is simple enough (e.g. $y'' + y = 1$).

  Of course if asked to use a method, you should used the method.
Topics

Below is a list of important topics to study. Guide to reading the list:

**Blue text** indicates critical topics - you should focus on these core ideas and methods first, as they also inform all the rest of the details.

**Red text** indicates topics that are more ‘specific’. They will only be a small part of the exam (if they appear at all), and they do not connect to other topics.

**Purple text** denotes details that are a common source of mistakes or confusion. Topics in black are somewhere in between.

The innemost dashed (−) notes are details about the topic; they may not explicit list some of the fundamental ideas implied in the circle-headings.

Core concepts

- **Fundamentals:** There are some fundamental concepts that appear throughout the course. You should understand these ideas and how they relate to each topic.
  - Homogeneous/inhomogeneous problems
    - How this relates to linearity (see below)
    - What it means to be homogeneous
    - How to recognize/handle inhomogeneous terms (details are topic specific)
    - Particular solutions (why use them)
  - Linearity
    - Linear operators (involving derivatives)
    - When is an operator linear? (define; show it directly)
    - Definition of a linear ODE/PDE \((L[y] = f)\)
    - Superposition of solutions; using it to break problems into simpler parts
    - Bases, linear independence
    - Idea of solving linear equations by writing solutions in terms of a basis
  - Types of solutions
    - What does it mean to be a solution? (answer depends on the topic)
    - Role of initial conditions and boundary conditions
    - Uniqueness of solutions
Difference between initial value problems and boundary value problems for ODEs (e.g. compare Chapter 3 with Chapter 11)

○ What can we say about solution behavior?
  - What do solutions do as $t \to \infty$ (long-term behavior)?
  - Oscillations, decay and growth
  - Limitations of methods (what we don’t get from the method)

○ Eigenfunctions (has its own list further down)

Some important topics

• **Fourier series:** You should have a solid understanding of the underlying ideas here, although they are really just a special case of eigenfunction/value problems more generally (see ‘Eigenfunctions’). It is **essential** that you are comfortable with computing Fourier series and sine/cosine series and using them to solve PDEs.
  - Calculating Fourier series and sine/cosine series
  - Convergence (the theorem for pointwise convergence)
  - Even and odd symmetry
  - Handling the $n = 0$ term correctly

• **Eigenfunctions:** As with Fourier series, it is essential that you can write functions in terms of an eigenfunction basis (i.e. solve for the coefficients). The basic theory is also important here.
  - Meaning of a basis for $L^2[a,b]$, orthogonality
    - Definition of the inner product, basic properties
    - orthonormal bases
  - Calculating eigenfunctions/values
    - How to show there are no eigenvalues in a certain range
    - Estimating eigenvalues
  - Eigenfunction expansions (coefficients)
  - Self-adjoint operators
    - Definition
    - Know how to check if an operator is self-adjoint
    - Showing a BVP is self-adjoint (write it as $L[y] = 0$ and show $L$ is self-adjoint)
  - Main result of Sturm-Liouville theory (solution structure for SLPs)
ODEs

- **Basic ODEs:** You should be comfortable with the solution process - be able to solve them *quickly* (and correctly), especially the ones highlighted in blue.
  - First-order, linear (integrating factors!)
  - Separable equations
  - Exact equations
  - Substitution methods (changing variables to get a nicer equation; the process is important but the specific substitutions are not)
  - Second-order, constant-coefficient ODEs
    - Linear independence of basis solutions
    - Homogeneous case (you **must** be able to solve these efficiently!)
    - Undetermined coefficients (be able to solve the ‘easy’ cases efficiently; the table of guesses will be given to you)
    - Variation of parameters (you will have the formula given; just know how to use it)
- **Laplace Transform:** The suggestions/topics for the second midterm apply to the final as well; it should be relatively straightforward.
  - Using the Laplace transform to solve IVPs
    - Writing discontinuous functions in terms of step functions
    - IVPs and transforms relating to δ’s, step functions
    - Use of convolutions
  - Calculating transforms and inverse transforms
    - (important) know how/when to use the rules in the table (except 4)
    - Know how to derive everything on the table except the convolution formula
- **Series solutions:** There is not much to know for power series solutions beyond the process. For regular singular points, getting the first term is the most important; the full Frobenius series, technicalities with repeated roots etc. are not important.
  - Computing radius of convergence (ratio test)
  - Power series solutions
    - What does it mean to be a series solution?
    - Know how to solve (reasonably simple) recurrences
Around points other than $x = 0$

- Regular singular points
  - Identifying singular points (regular/irregular)
  - *Note: further details are not worth studying.*

**PDEs:**

Several PDEs were introduced but all of them can be solved in the same way. Separation of variables (which you should know well because it is a good tool for computing solutions) and the eigenfunction method are both essential. You also need to know the various tricks for reducing problems to simpler parts (superposition, steady states).

There is much less to know on the level of theory for PDEs for this course (existence, uniqueness, etc.) besides what is needed to get a solution; most of what we covered was done by solving the PDE and interpreting the solution directly. It is important to understand how these problems are set up (boundary conditions etc.). A knowledge of how solutions behave (oscillations, exponential decay etc.) can be useful intuition but getting the solution is more important for the exam.

- Heat equation, wave equation, Laplace’s equation
  - Boundary/initial conditions for each
  - General behavior of solutions for each type of equation
  - Homogeneous vs. inhomogeneous PDEs
  - How linearity is relevant to solving linear PDEs
  - Deriving/solving the eigenvalue problem

- Separation of variables
  - Extracting the constraints on the separated parts from the BCs (e.g. plugging in $u = X(x)T(t)$ into $u(0, t) = 0$ to conclude that $X(0) = 0$)
  - Putting together solutions; applying initial/boundary condition to get coefficients

- More solution techniques
  - Using a steady state to get rid of a source term
  - Splitting a problem into parts with superposition (know the concept and also be able to use it e.g. steady states above; for the wave equation; for Laplace’s equation)
  - Moving a boundary condition to a source with a particular solution

- Solution using eigenfunctions
Solving inhomogeneous BVPs with eigenfunctions (Chapter 11.3) (the method is the same as the one for PDEs so you should know how to do this in theory, but the details are not important.)

For PDE problems, finding the eigenfunctions (see above; also Sturm-Liouville theory and Fourier series)

– Note that the eigenfunctions satisfy the homogeneous BCs; e.g. if \( u(0, t) = 1 \) then \( \phi(0) = 0 \), not \( \phi(0) = 1 \)

PDEs with source terms (the easier case)

PDEs with inhomogeneous BCs (the harder case; use only if necessary)