The homogeneous constant-coefficient ordinary differential equation

\[ \sum_{j=0}^{n} a_j y^{(j)} = 0 \]

is such that for any \( r \)

\[ \sum_{j=0}^{n} a_j \frac{d^j e^{rx}}{dx^j} = e^{rx} \sum_{j=0}^{n} a_j r^j \]

This suggests that we define and factor the \textbf{characteristic polynomial}

\[ \sum_{j=0}^{n} a_j r^j = a_n (r - r_1) \cdots (r - r_n) \]

where \( r_1, \ldots, r_n \) are the roots of the characteristic polynomial, and possibly complex. This means that we can write the differential equation in the form

\[ 0 = \{ \sum_{j=0}^{n} a_j D^j \} y = a_n (D - r_1) \cdots (D - r_n) y \]

For any integer \( k \) and any polynomial \( p \in \mathcal{P}_{k-1} \)

\[ D^k p = 0 \]

In other words, the general solution to \( y^{(k)} = 0 \) is an arbitrary polynomial of degree \( k - 1 \).

Note that

\[ (D - r)\{ e^{rx} y(x) \} = (e^{rx} y)'(x) - re^{rx} y(x) = e^{rx} y'(x) + re^{rx} y(x) - r e^{rx} y(x) = e^{rx} Dy(x) \]

It follows that

\[ (D - r)^k \{ e^{rx} y(x) \} = e^{rx} D^k y(x) \]

Thus the general solution of \( (D-r)^k z(x) = 0 \) is \( z(x) = e^{rx} \mathcal{P}_{k-1} \), or \( e^{rx} \) times an arbitrary polynomial of degree \( k - 1 \).

Suppose that we want to solve the inhomogeneous equation

\[ D^n y = p(x) \in \mathcal{P}_k \]

Then \( y \in \mathcal{P}_{n+k} \) can be found either by integrating \( p \) for \( n \) times, or by trying a solution of the form

\[ y_p(x) = \sum_{j=0}^{n+k} c_j x^j \]

and solving for the \( c_j \)'s. The latter approach is called \textbf{undetermined coefficients}. 
Suppose that we want to solve the inhomogeneous equation
\[(D - r)^n y = e^{rx} p(x) \in e^{rx} P_k\]
Then \(y \in P_{n+k}\) can be found by trying a solution of the form
\[y_p(x) = e^{rx} \sum_{j=0}^{n+k} c_j x^j\]
and solving for the \(c_j\)’s. Note that the order of the polynomial in the solution is the sum of the multiplicity of the root \(r\) of the differential equation, and the order of the polynomial \(p\) in the right-hand side of the differential equation.
If \(p(x) \in P_k\) and \(s \neq r\), then
\[(D - s)\{e^{rx} p(x)\} = re^{rx} p(x) + e^{rx} p'(x) - se^{rx} p(x) = e^{rx}\{(r - s)p(x) + p'(x)\} \in e^{rx} P_k\]
It follows that for any \(n \geq 0\)
\[(D - s)^{n}\{e^{rx} p(x)\} \in e^{rx} P_k\]
Suppose that we want to solve
\[(D - r_1)^{n_1} \ldots (D - r_t)^{n_t} y = e^{rx} p(x) \in e^{rx} P_k\]
Then our results above suggest that we should try
\[y_p(x) = e^{rx} q(x) \in e^{rx} P_{n_1+k}\]
This is because the operators \((D - r_j)^{n_j}\) for \(j > 1\) will leave \(y\) in the same set \(e^{rx} P_{n_1+k}\), and \((D - r_1)^{n_1}\) will move \(y\) to \(e^{rx} P_k\). Note that the order of the roots \(r_j\) in the factored form of the differential equation does not matter.
Here are some examples:

#7 p. 211 \(D(D+1)y = y'' + y' = e^x + \cos(2x) - x\). The particular solution has the form \(y_p(x) = ae^x + b \sin(2x) + c \cos(2x) + d_2 x^2 + d_1 x + d_0\). This is because the roots of the characteristic polynomial are 0 and -1, and the right-hand side of the differential equation involves exponentials with exponents 1x, 2x, -2x and 0x. The factors \(D\) and \(D+1\) preserve the first 3 exponents, and the factor \(D\) reduces the order of the terms with 0x exponent.

#24 p. 211 \((D - 2)(D + 4)^2 y = y'''' + 6y''' - 32y = 3xe^{2x} - 7e^{-4x}\). The particular solution has the form \(y_p(x) = (a_2 x^2 + a_1 x + a_0)e^{2x} + (b_2 x^2 + b_1 x + b_0)e^{-4x}\). This is because the roots of the characteristic polynomial are 2 and -4 (with multiplicity 2), and the right-hand side of the differential equation involves exponentials with exponents 2 and -4.

#23 p. 211 \((D + i)(D - i)y = y'' + y = 2x^2 + 2x \cos x = 2x^2 + x(e^{ix} + e^{-ix})\). The particular solution has the form \(y_p(x) = a_2 x^2 + a_1 x + a_0 + (b_2 x^2 + b_1 x + b_0) \cos x + (c_2 x^2 + c_1 x + c_0) \sin x\).

#24 p. 211 \((D - 2)(D + 4)^2 y = y'''' + 6y''' - 32y = 7xe^{2x} - 7e^{-4x}\). The particular solution has the form \(y_p(x) = (a_2 x^2 + a_1 x + a_0)e^{2x} + (b_2 x^2 + b_1 x + b_0)e^{-4x}\).

#26 p. 211 \((D + 2i)^2(D - 2i)^2 y = y'''' + 8y''' + 16y = 4x - 2e^{4x} - 4x \sin 2x = 4x - 2e^{4x} - \cos 4x + 2x(e^{2ix} - e^{-2ix})/2i\). The particular solution has the form \(y_p(x) = (a_1 x + a_0) + b_0 e^{4x} + c_0 \cos 4x + (d_3 x^3 + d_2 x^2 + d_1 x + d_0) \cos 2x + (e_3 x^3 + e_2 x^2 + e_1 x + e_0) \sin 2x\).