Math 108 Eigenfunction Expansions
November 4, 2006

Eigenfunction expansions can be used to solve partial differential equations, such as the heat equation and the wave equation. In particular, we can use eigenfunction expansions to treat boundary conditions with inhomogeneities that change in time, or partial differential equation inhomogeneities that change in time. The technique will be to use the Lagrange identity (on p. 667 of the book) to develop ordinary differential equations for the coefficients in the eigenfunction expansion. We will examine this approach through several examples, then explain the general idea.

For our first example, let us consider an easy problem that we previously solved by separation of variables.

Problem 7 on page 610:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - 100\frac{\partial^2 u}{\partial x^2} = 0 \ , \ 0 < x < 1 \ , \ 0 < t \\
&u(0,t) = 0 \ , \ 0 < t \\
&u(1,t) = 0 \ , \ 0 < t \\
&u(x,0) = \sin(2\pi x) - \sin(5\pi x) \ , \ 0 < x < 1
\end{aligned}
$$

The easy way to solve this equation is to use separation of variables, as in section 10.5, and that is the approach we would suggest that you use to solve this problem on a test. *Separation of variables works because the differential equation and the boundary conditions are all homogeneous. That is what allows us to take a linear combination of functions $X_n(x)T_n(t)$ satisfying the (homogeneous) boundary conditions, and get a general function satisfying the (homogeneous) boundary conditions.* Below, we will show how the eigenfunction expansion approach works.

First, let us motivate the determination of a Sturm-Liouville problem for this heat equation. We would find that a solution of the heat equation of the form $X_n(x)T_n(t)$ would satisfy

$$\frac{T_n'}{T_n} = \frac{100X_n''}{X_n} = -\lambda_n$$

with

$$X_n(0) = 0 = X_n(1) \ .$$

The associated Sturm-Liouville problem is

$$-(100X_n')' = \lambda_n X_n \ , \ X_n(0) = 0 = X_n(1) \ .$$

In the book's description of Sturm-Liouville problems on p. 666, we have $p(x) = 1, q(x) = 0$ and $r(x) = 1$. We solve the differential equation for the eigenfunctions to get

$$X_n(x) = \sin(n\pi x) \ , \ \lambda_n = (10n\pi)^2 \ .$$

Next, let us solve the original partial differential equation by using eigenfunction expansions. Theorem 11.2.4 on p. 672 of the book shows us that we can write

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x)$$

for some coefficients $c_n(t)$ that are yet to be determined. We will develop ordinary differential equations to determine $c_n(t)$. The orthogonality of the eigenfunctions implies that

$$\int_0^1 u(x,t)r(x)X_m(x)\,dx \equiv (u, rX_m) = \sum_{n=1}^{\infty} c_n(t)\,(X_n, rX_m) = c_m(t)\,(X_m, rX_m)$$
$$\equiv c_m(t)\int_0^1 r(x)X_m(x)^2\,dx$$

We note that $r(x) = 1$ in this example, and solve for $c_m(t)$ to get

$$c_m(t) = \frac{\int_0^1 u(x,t)X_m(x)\,dx}{\int_0^1 X_m(x)^2\,dx}\,. \tag{1}$$

This does not solve the problem, because we do not know what $u(x,t)$ is, so we can't compute $c_m(t)$ directly from this formula. However, we can use this equation to determine an ordinary differential equation for $c_m(t)$.

The important facts in developing this ordinary differential equation are that $u(x,t)$ solves the heat equation and satisfies boundary conditions (that are not necessarily homogeneous in general, but are homogeneous in this example), and $X_m(x)$ solves the associated Sturm-Liouville problem.

Let us replace m with n, since the latter is our summation variable.

We begin by rewriting equation (1)

$$\lambda_n c_n(t)\int_0^1 X_n(x)^2\,dx = \lambda_n\int_0^1 u(x,t)X_n(x)\,dx$$

then we use the Sturm-Liouville problem for X_n to write

$$= \int_0^1 u(x,t)\left\{-100X_n''(x)\right\}\,dx$$

then we integrate by parts and use the boundary conditions for u to get

$$= -\left[u(x,t)100X_n'(x)\right]_0^1 + \int_0^1 \frac{\partial u}{\partial x}(x,t)100X_n'(x)\,dx$$
$$= \int_0^1 \frac{\partial u}{\partial x}(x,t)100X_n'(x)\,dx$$

then we integrate by parts again and use the (homogeneous) boundary conditions for X_n to get

$$= \left[100\frac{\partial u}{\partial x}(x,t)X_n(x)\right]_0^1 - \int_0^1\left\{100\frac{\partial^2 u}{\partial x^2}(x,t)\right\}X_n(x)\,dx$$
$$= -\int_0^1\left\{100\frac{\partial^2 u}{\partial x^2}(x,t)\right\}X_n(x)\,dx$$

then we use the heat equation to get

$$= -\int_0^1 \frac{\partial u}{\partial t}(x,t) X_n(x)\, dx = -\frac{d}{dt}\int_0^1 u(x,t) X_n(x)\, dx$$

then we use equation (1) to write

$$= -c_n'(t) \int_0^1 X_n(x)^2\, dx\ .$$

We can cancel out $\int_0^1 X_n(x)^2\, dx$ to get

$$-c_n'(t) = \lambda_n c_n(t)\ .$$

The initial values $c_n(0)$ are determined by the initial condition for the heat equation:

$$\sin(2\pi x) - \sin(5\pi x) = u(x,0) = \sum_{n=1}^{\infty} c_n(0) \sin(n\pi x)$$

implies that $c_2(0) = 1$, $c_5(0) = -1$ and $c_n(0) = 0$ otherwise.

Note that in this case, $c_n(t)$ satisfies the same differential equation as the separation of variables function $T_n(t)$. Separation of variables is easier to use, but works for this problem only because the boundary conditions are homogeneous.

Our next example involve a homogeneous partial differential equation with an inhomogeneous boundary condition.

Problem 7 on page 620:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \ , \ 0 < x < L \ , \ 0 < t \\
&\frac{\partial u}{\partial x}(0,t) - u(0,t) = 0 \ , \ 0 < t \\
&u(L,t) = T \ , \ 0 < t \\
&u(x,0) = u_0(x) \ , \ 0 < x < L
\end{aligned}$$

We could solve this problem by finding a steady-state solution $w(x)$ and subtracting it from $u(x,t)$ to find equations for the transient, as described in section 10.6 of the book. This is the approach we recommend, if you should find this problem on a test.

Let us illustrate the eigenfunction approach for this problem. We cannot use separation of variables directly on this problem, because the right-hand boundary condition is inhomogenous. However, the Sturm-Liouvile problem for this heat equation is

$$\begin{aligned}
&-\alpha^2 \left(X_n'\right)' = \lambda_n X_n \\
&X_n'(0) - X_n(0) = 0 \\
&X_n(L) = 0
\end{aligned}$$

This is what separation of variables would have found, if the boundary conditions were homogeneous. Here $p(x) = 1, q(x) = 0$ and $r(x) = 1$. These equations imply that

$$X_n(x) = \frac{\sqrt{\lambda_n}}{\alpha} \cos(x\sqrt{\lambda_n}/\alpha) + \sin(x\sqrt{\lambda_n}/\alpha)$$

where

$$\sqrt{\lambda_n}/\alpha = -\tan(L\sqrt{\lambda_n}/\alpha) \ .$$

We can write

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x)$$

for some coefficients $c_n(t)$ that are yet to be determined. Again, the orthogonality of the eigenfunctions implies that

$$\int_0^L u(x,t) X_n(x) \, dx = c_n(t) \int_0^L X_n(x)^2 \, dx$$

We will use this equation to determine an ordinary differential equation for $c_n(t)$.

We start with the equation for $c_n(t)$ from the orthogonality of the eigenfunctions

$$\lambda_n c_n(t) \int_0^L X_n(x)^2 \, dx = \int_0^L u(x,t) \left\{\lambda_n X_n(x)\right\} \, dx$$

then we use the Sturm-Liouville equation to get

$$= \int_0^L u(x,t) \left\{ -\alpha^2 X_n''(x) \right\} \, dx$$

then we integrate by parts and use the boundary conditions on u

$$\begin{aligned}
&= - \left[u(x,t)\alpha^2 X_n'(x) \right]_0^L + \int_0^L \frac{\partial u}{\partial x}(x,t)\alpha^2 X_n'(x) \, dx \\
&= -T X_n'(L) + \frac{\partial u}{\partial x}(0,t) X_n'(0) + \int_0^L \frac{\partial u}{\partial x}(x,t)\alpha^2 X_n'(x) \, dx
\end{aligned}$$

then we integrate by parts again and use the boundary conditions on X_n

$$\begin{aligned}
&= -T\alpha^2 X_n'(L) + \frac{\partial u}{\partial x}(0,t)\alpha^2 X_n'(0) + \left[\alpha^2 \frac{\partial u}{\partial x}(x,t) X_n(x) \right]_0^L \\
&- \int_0^L \left\{ \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) \right\} X_n(x) \, dx \\
&= -T\alpha^2 X_n'(L) - \int_0^L \left\{ \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) \right\} X_n(x) \, dx
\end{aligned}$$

then we use the heat equation to write

$$\begin{aligned}
&= -T\alpha^2 X_n'(L) - \int_0^L \frac{\partial u}{\partial t}(x,t) X_n(x) \, dx \\
&= -T\alpha^2 X_n'(L) - \frac{d}{dt} \int_0^L u(x,t) X_n(x) \, dx
\end{aligned}$$

then we use the orthogonality equation for $c_n(t)$ to get

$$= -T\alpha^2 X_n'(L) - c_n'(t) \int_0^L X_n(x)^2 \, dx$$

In other words, we have the ordinary differential equation

$$c_n'(t) = -\lambda_n c_n(t) - \frac{T\alpha^2 X_n'(L)}{\int_0^L X_n(x)^2 \, dx}$$

The initial condition

$$u_0(x) = u(x,0) = \sum_{n=1}^{\infty} c_n(0) X_n(x)$$

implies that

$$c_n(0) = \frac{\int_0^L u_0(x) X_n(x) \, dx}{\int_0^L X_n(x)^2 \, dx}$$

The ordinary differential equation involves the known eigenfunctions $X_n(x)$, the known initial temperature distribution $u_0(x)$ and the known boundary temperature T.

We can solve the ordinary differential equation for $c_n(t)$ to get

$$c_n(t) = e^{-\lambda_n t} c_n(0) - (1 - e^{-\lambda_n t}) \frac{T\alpha^2 X_n'(L)}{\lambda_n \int_0^L X_n(x)^2 \, dx}$$

The solution of the original problem is now solved as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x)$$

The next heat equation is inhomogeneous, with homogeneous boundary conditions.

Problem 22 on page 690:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^{-t}(1-x) \ , \ 0 < x < 1 \ , \ 0 < t \\
&u(0,t) = 0 \ , \ 0 < t \\
&\frac{\partial u}{\partial x}(1,t) = 0 \ , \ 0 < t \\
&u(x,0) = 0 \ , \ 0 < x < 1
\end{aligned}$$

Since the right-hand side of the differential equation is a function of both x and t, eigenfunction expansions are the only way we have to solve this problem.

The associated Sturm-Liouville problem is

$$\begin{aligned}
&-X_n'' = \lambda_n X_n(x) \ , \ 0 < x < 1 \\
&X_n(0) = 0 \\
&X_n'(1) = 0
\end{aligned}$$

Here $p(x) = 1, q(x) = 0$ and $r(x) = 1$. The eigenfunctions are

$$X_n(x) = \sin(x\sqrt{\lambda_n})$$

and the eigenvalues are

$$\lambda_n = (\frac{2n-1}{2}\pi)^2 \ .$$

The solution of the heat equation can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x) \ .$$

Orthogonality of the eigenfunctions implies that

$$c_n(t) = \frac{\int_0^1 u(x,t) X_n(x) \, dx}{\int_0^1 X_n(x)^2 \, dx} \tag{2}$$

We will use the heat equation for u, Sturm-Liouville problem for X_n and boundary conditions for both to determine an ordinary differential for $c_n(t)$.

We begin by rewriting the orthogonality equation for $c_n(t)$,

$$\lambda_n c_n(t) \int_0^1 X_n(x)^2 \, dx = \int_0^1 u(x,t) \left\{\lambda_n X_n(x)\right\} \, dx$$

then we use the Sturm-Liouville problem for X_n

$$= \int_0^1 u(x,t)\left\{-X_n(x)''\right\}\ dx$$

then we integrate by parts and use the boundary conditions

$$= -\left[u(x,t)X_n'(x)\right]_0^1 + \int_0^1 \frac{\partial u}{\partial x}(x,t)X_n(x)'\ dx = \int_0^1 \frac{\partial u}{\partial x}(x,t)X_n(x)'\ dx$$

then we integrate by parts again and use the boundary conditions

$$= \left[\frac{\partial u}{\partial x}(x,t)X_n(x)\right]_0^1 - \int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)X_n(x)\ dx = -\int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)X_n(x)\ dx$$

then we use the heat equation

$$\begin{aligned} &= -\int_0^1 \left\{\frac{\partial u}{\partial t}(x,t) - e^{-t}(1-x)\right\} X_n(x)\ dx \\ &= -\frac{d}{dt}\int_0^1 u(x,t)X_n(x)\ dx + \int_0^1 e^{-t}(1-x)X_n(x)\ dx \end{aligned}$$

then we use the orthogonality equation for $c_n(t)$

$$= -c_n'(t)\int_0^1 X_n(x)^2\ dx + \int_0^1 e^{-t}(1-x)X_n(x)\ dx$$

We can rewrite this result as

$$c_n'(t) = -\lambda_n c_n(t) + \frac{\int_0^1 e^{-t}(1-x)X_n(x)\ dx}{\int_0^1 X_n(x)^2\ dx}$$

Since $X_n(x)$ is a known function, we can compute the integrals in this ordinary differential equation; we leave this to the student. The initial value $c_n(0)$ comes from the initial condition on u:

$$c_n(0) = \frac{\int_0^1 u(x,0)X_n(0)\ dx}{\int_0^1 X_n(x)^2\ dx} = 0\ .$$

The ordinary differential equation for $c_n(t)$ involves the known eigenfunction $X_n(x)$ and the known inhomogeneity in the heat equation. We leave the details of computing the coefficients in this ordinary differential equation to the students.

After we solve these ordinary differential equations for $c_n(t)$, the solution of the inhomogeneous heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)X_n(x)\ .$$

Next, let us consider a dispersive wave equation.

Problem 23, page 637:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + (\gamma a)^2 u = 0 \ , \ 0 < x < L \ , \ 0 < t \\
&u(0,t) = 0 \ , \ 0 < t \\
&u(L,t) = 0 \ , \ 0 < t \\
&u(x,0) = u_0(x) \ , \ 0 < x < L \\
&\frac{\partial u}{\partial t}(x,0) = 0 \ , \ 0 < x < L
\end{aligned}
$$

This problem involves a homogeneous wave equation with homogeneous boundary conditions. It could be solved either by separation of variables, or by an eigenfunction expansion. We would recommend using separation of variables to solve this problem on a test, but we will show how to use eigenfunction expansions below.

The Sturm-Liouville problem for this wave equation is

$$
\begin{aligned}
&-a^2 X_n'' + (\gamma a)^2 X_n = \lambda_n X_n(x) \ , \ 0 < x < L \\
&X_n(0) = 0 \\
&X_n(L) = 0 \ .
\end{aligned}
$$

Here $p(x) = a^2, q(x) = (\gamma a)^2$ and $r(x) = 1$ in the book's definition of a Sturm-Liouville problem. These equations imply that the eigenfunctions are

$$X_n(x) = \sin(n\pi x)$$

and the eigenvalues are

$$\lambda_n = a^2[\gamma^2 + (n\pi/L)^2] \ .$$

We write the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x)$$

where orthogonality of the eigenfunctions implies that

$$c_n(t) = \frac{\int_0^L u(x,t) X_n(x) \ dx}{\int_0^L X_n(x)^2 \ dx}$$

We will develop an ordinary differential equation for $c_n(t)$.

We begin by rewriting the orthogonality equation

$$\lambda_n c_n(t) \int_0^L X_n(x)^2 \ dx = \int_0^L u(x,t) \left\{ \lambda_n X_n(x) \right\} \ dx$$

then we use the Sturm-Liouville problem for X_n to write

$$= \int_0^L u(x,t)\left\{-a^2 X_n''(x) + (\gamma a)^2 X_n(x)\right\}\ dx$$

then we integrate by parts and use boundary conditions to write

$$\begin{aligned}
&= -\left[u(x,t)a^2 X_n'(x)\right]_0^L + \int_0^L \frac{\partial u}{\partial x}(x,t)a^2 X_n'(x) + u(x,t)(\gamma a)^2 X_n(x)\ dx \\
&= \int_0^L \frac{\partial u}{\partial x}(x,t)a^2 X_n'(x) + u(x,t)(\gamma a)^2 X_n(x)\ dx
\end{aligned}$$

then we integrate by parts again and use boundary conditions to write

$$\begin{aligned}
&= \left[\frac{\partial u}{\partial x}(x,t)a^2 X_n(x)\right]_0^L + \int_0^L \left\{-a^2\frac{\partial^2 u}{\partial x^2}(x,t) + (\gamma a)^2 u(x,t)\right\} X_n(x)\ dx \\
&= \int_0^L \left\{-a^2\frac{\partial^2 u}{\partial x^2}(x,t) + (\gamma a)^2 u(x,t)\right\} X_n(x)\ dx
\end{aligned}$$

then we use the wave equation to write

$$= \int_0^L \frac{\partial^2 u}{\partial t^2}(x,t)X_n(x)\ dx = \frac{d^2}{dt^2}\int_0^L u(x,t)X_n(x)\ dx$$

then we use the orthogonality equation to write

$$= c''(t)\int_0^L X_n(x)^2\ dx$$

The initial conditions for $c_n(t)$ come from the initial conditions for $u(x,t)$ and the orthogonality equation:

$$\begin{aligned}
c_n(0) &= \frac{\int_0^L u(x,0)X_n(x)\ dx}{\int_0^L X_n(x)^2\ dx} = \frac{u_0(x)X_n(x)\ dx}{\int_0^L X_n(x)^2\ dx} \\
c_n'(0) &= \frac{\int_0^L \frac{\partial u}{\partial t}(x,0)X_n(x)\ dx}{\int_0^L X_n(x)^2\ dx} = 0\ .
\end{aligned}$$

After we solve these ordinary differential equations for $c_n(t)$, the solution of the original problem is determined as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)X_n(x)\ .$$

Now we can describe the general process for solving time-dependent partial differential equations by eigenfunction expansions. Consider the partial differential equation

$$r(x)L_t(u) + L_x(u) = f(x,t) \ , \ a < x < b \ , \ 0 < t < \tau \tag{3a}$$

$$k_a \frac{\partial u}{\partial x}(a,t) + h_a u(a,t) = g_a(t) \tag{3b}$$

$$k_b \frac{\partial u}{\partial x}(b,t) + h_b u(b,t) = g_b(t) \tag{3c}$$

plus initial conditions that are appropriate for the operator L_t. We will assume that both L_t and L_x are linear operators on their arguments u. Further, we will assume that L_x is self-adjoint.

For example, the heat equation

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

takes $r(x) = 1$, $L_t(u) = \frac{\partial u}{\partial t}$ and $L_x(u) = -\alpha^2 \frac{\partial^2 u}{\partial x^2}$. An appropriate initial condition would be $u(x,0) = u_0(x)$. The Sturm-Liouville problem corresponding to L_x would be $-\alpha^2 X_n'' = \lambda_n X_n$. In this case, $p(x) = 1$ and $q(x) = 0$.

As another example, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

which takes $r(x) = 1$, $L_t(u) = \frac{\partial^2 u}{\partial t^2}$ and $L_x(u) = -c^2 \frac{\partial^2 u}{\partial x^2}$. Appropriate initial conditions would be $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$. The Sturm-Liouville problem corresponding to L_x would be $-c^2 X_n'' = \lambda_n X_n$. In this case, $p(x) = c^2$ and $q(x) = 0$.

A final example is given by Laplace's equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

which takes $r(x) = 1$, $t = y$, $L_y(u) = -\frac{\partial^2 u}{\partial y^2}$ and $L_x(u) = -\frac{\partial^2 u}{\partial x^2}$. Appropriate boundary conditions in y might be $u(x,0) = u_0(x)$ and $u(x,c) = v_0(x)$ for some $c > 0$. The Sturm-Liouville problem corresponding to L_x would be $-X_n'' = \lambda_n X_n$. In this case, $p(x) = 1$ and $q(x) = 0$.

Separated solutions of the homogeneous partial differential equation (3a) with homogeneous boundary conditions (*i.e.*, with $f = 0$, $g_a = 0$ and $g_b = 0$) have the form $X_n(x)T_n(t)$ where

$$r(x)X_n(x)L_t(T_n) = L_x(X_n)T_n(t)$$

After dividing both sides by $r(x)X_n(x)T_n(t)$ we get

$$\frac{L_x(X_n)}{r(x)X_n(x)} = \frac{L_t(T_n)}{T_n} = \lambda_n$$

for some constant λ_n. This plus the homogeneous forms of (3b) and (3c) lead to an eigenvalue problem for X_n:

$$L_x(X_n) = \lambda_n r(x) X_n(x) = 0 \tag{4a}$$

$$k_a X_n'(a) + h_a X_n(a) = 0 \tag{4b}$$

$$k_b X_n'(b) + h_b X_n(b) = 0 \tag{4c}$$

If

$$L_x(u) = -\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu$$

where p is nonzero, and q does not have the opposite sign of p and r is either positive or negative for all x, then the functions $X_n(x)$ are eigenfunctions of a Sturm-Liouville operator. This means that we can write the solution $u(x,t)$ of the original differential equation in the form of an eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) X_n(x) \tag{5}$$

for some coefficients $c_n(t)$. Let us discover an ordinary differential equation for these coefficients $c_n(t)$.

Since eigenfunctions of Sturm-Liouville operators are orthogonal, meaning that

$$\int_a^b r(x) X_n(x) X_m(x)\, dx = 0 \text{ for } n \neq m$$

we have

$$\int_a^b r(x) u(x,t) X_m(x)\, dx = \sum_{n=1}^{\infty} c_n(t) \int_a^b r(x) X_n(x) X_m(x)\, dx = c_m(t) \int_a^b r(x) X_m(x)^2\, dx$$

We can solve this equation for $c_m(t)$:

$$c_m(t) = \frac{\int_a^b r(x) u(x,t) X_m(x)\, dx}{\int_a^b r(x) X_m(x)^2\, dx} \tag{6}$$

We will use this equation to find an ordinary differential equation for $c_m(t)$.

We begin with the orthogonality equation for $c_n(t)$

$$\lambda_n c_n(t) \int_a^b r(x) X_n(x)^2\, dx = \int_a^b u(x,t) \left\{\lambda_n r(x) X_n(x)\right\}\, dx$$

then we use the Sturm-Liouville equation

$$= \int_a^b u(x,t) \left\{L_x(X_n)(x)\right\}\, dx$$
$$= \int_a^b u(x,t) \left\{-\left(p(x) X_n'(x)\right)' + q(x) X_n(x)\right\}\, dx$$

then we integrate by parts

$$= -\left[u(x,t) p(x) X_n'(x)\right]_a^b + \int_a^b \frac{\partial u}{\partial x}(x,t) p(x) X_n'(x) + u(x,t) q(x) X_n(x)\, dx$$

and integrate by parts again

$$
\begin{aligned}
&= -\left[u(x,t)p(x)X_n'(x)\right]_a^b + \left[\frac{\partial u}{\partial x}(x,t)p(x)X_n(x)\right]_a^b \\
&+ \int_a^b \left\{-\frac{\partial}{\partial x}\left(p(x)\partial u\partial x(x,t)\right) + q(x)u(x,t)\right\} X_n(x)\, dx \\
&= p(x)\left[-u(x,t)X_n'(x) + \frac{\partial u}{\partial x}(x,t)X_n(x)\right]_a^b + \int_a^b \left\{L_x(u)(x,t)\right\} X_n(x)\, dx
\end{aligned}
$$

then we use the partial differential equation for u

$$
\begin{aligned}
&= p(x)\left[-u(x,t)X_n'(x) + \frac{\partial u}{\partial x}(x,t)X_n(x)\right]_a^b \\
&+ \int_a^b \left\{-L_t(u)(x,t) + f(x,t)\right\} X_n(x)\, dx \\
&= p(x)\left[-u(x,t)X_n'(x) + \frac{\partial u}{\partial x}(x,t)X_n(x)\right]_a^b + \int_a^b f(x,t)X_n(x)\, dx \\
&- L_t\left(\int_a^b u(x,t)X_n(x)\, dx\right)(t)
\end{aligned}
$$

then we use the orthogonality equation for c_n

$$
\begin{aligned}
&= p(x)\left[-u(x,t)X_n'(x) + \frac{\partial u}{\partial x}(x,t)X_n(x)\right]_a^b + \int_a^b f(x,t)X_n(x)\, dx \\
&- L_t(c_n)(t)\int_a^b r(x)X_n(x)^2\, dx
\end{aligned}
$$

This gives us an ordinary differential equation for c_n:

$$
L_t(c_n)(t) = -\lambda_n c_n(t) + \frac{\int_a^b f(x,t)X_n(x)\, dx}{\int_a^b r(x)X_n(x)^2\, dx} + \frac{p(x)\left[-u(x,t)X_n'(x) + \frac{\partial u}{\partial x}(x,t)X_n(x)\right]_a^b}{\int_a^b r(x)X_n(x)^2\, dx} \tag{7}
$$

The evaluation of the terms on the boundary depends on the boundary conditions. Suppose that $k_b \neq 0$ in (3c). Then

$$
\begin{aligned}
p(b)\left[u(b,t)X_n'(b) - \frac{\partial u}{\partial x}(b,t)X_n(b)\right] &= -\frac{p(b)}{k_b}\left[h_b u(b,t) + k_b\frac{\partial u}{\partial x}(b,t)\right] X_n(b) \\
&= -\frac{p(b)}{k_b} g_b(t)X_n(b)
\end{aligned}
$$

involves known quantities. On the other hand, if $k_b = 0$ then we must have $h_b \neq 0$ and $X_n(b) = 0$, so

$$
p(b)\left[u(b,t)X_n'(b) - \frac{\partial u}{\partial x}(b,t)X_n(b)\right] = -\frac{p(b)}{h_b} g_b(t)X_n'(b)
$$

involves known quantities. The boundary condition at $x = a$ can be evaluated in a similar fashion.

In the case when $k_a \neq 0 \neq k_b$, equation (7) leads to the following problem for $c_n(t)$:

$$L_t(c_n) + \lambda_n c_n = \frac{-p(b)g_b(t)X_n(b)/k_b + p(a)g_a(t)X_n(a)/k_a + \int_a^b f(x,t)X_n(x)\,dx}{\int_a^b r(x)X_n(x)^2\,dx} \tag{8}$$

The initial or boundary condition(s) for c_n come from the initial or boundary conditions for u associated with t, and thus depend on the problem being solved. If $k_a = 0$ and $k_b \neq 0$, then $c_n(t)$ is the solution of

$$L_t(c_n) + \lambda_n c_n = \frac{-p(b)g_b(t)X_n'(b)/h_b + p(a)g_a(t)X_n(a)/k_a + \int_a^b f(x,t)X_n(x)\,dx}{\int_a^b r(x)X_n(x)^2\,dx} \tag{9}$$

If $k_a \neq 0$ and $k_b = 0$, then

$$L_t(c_n) + \lambda_n c_n = \frac{-p(b)g_b(t)X_n(b)/k_b + p(a)g_a(t)X_n'(a)/h_a + \int_a^b f(x,t)X_n(x)\,dx}{\int_a^b r(x)X_n(x)^2\,dx} \tag{10}$$

Finally, if $k_a = 0$ and $k_b = 0$, then

$$L_t(c_n) + \lambda_n c_n = \frac{-p(b)g_b(t)X_n'(b)/h_b + p(a)g_a(t)X_n'(a)/h_a + \int_a^b f(x,t)X_n(x)\,dx}{\int_a^b r(x)X_n(x)^2\,dx} \tag{11}$$