Anisotropic Harmonic Analysis and for Remotely Sensed Data

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Background on Harmonic Analysis: Classical and Anisotropic
Harmonic analysis provides methods for the decomposition of functions into simpler constituent atoms.

The nature of the decomposition is flexible, and the development of new methods is a major component of the field.

Given a function with certain known properties, a particular method of decomposition might be especially convenient.

Beyond being of theoretical interest, these decomposition can be used for applications.
The most classical decomposition system of the subject is Fourier series, with ideas dating back at least as far as Lagrange’s study of vibrating strings in the eighteenth century.

Suppose \( f \in L^2([0, 1]^d) \). Then \( f \) may be decomposed in the following manner, with convergence in the \( L^2([0, 1]^d) \) norm:

\[
f(x) = \sum_{m \in \mathbb{Z}^d} \left( \int_{[0, 1]^d} f(y) e^{-2\pi i \langle m, y \rangle} \, dy \right) e^{-2\pi i \langle m, x \rangle}.
\]

This method decomposes \( f \) with respect to its frequency content, as measured by the Fourier coefficients

\[
\int_{[0, 1]^d} f(y) e^{-2\pi i \langle m, y \rangle} \, dy.
\]
A different type of decomposition, based on scale and translation, was pioneered in the 1980s and 1990s \(^1,^2\).

Let \( f \in L^2(\mathbb{R}^2) \), and let \( \psi \) be a wavelet function. Then \( f \) may be decomposed in the following manner, with convergence in the \( L^2(\mathbb{R}^2) \) norm:

\[
 f = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^2} \langle f, \psi_{m,n} \rangle \psi_{m,n},
\]

where \( \psi_{m,n}(x) := |\det A|^m \psi(A^m x - n) \), \( A \in GL_2(\mathbb{R}) \). A typical choice for \( A \) is the dyadic isotropic matrix

\[
 A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\]

\(^1\) I. Daubechies. “Orthonormal bases of compactly supported wavelets.” Communications on pure and applied mathematics 41.7 (1988): 909-996.

Fourier methods proved fundamental in the early development of signal processing, and also in the study of physics.

Wavelets revolutionized the fields of image compression, fusion, and registration.

Both methods can be implemented with fast, efficient numerical algorithms in both low level (C) and high level (MATLAB) languages.

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These classical methods are known to be suboptimal for representing singularities in functions.

That is, if a function is singular (one dimension) or singular in a given direction (higher dimensions), these transforms fail to efficiently represent the singularity in the coefficients they generate.

For Fourier series, this is the Gibbs phenomenon: many Fourier coefficients are needed to accurately account for a discontinuity.
Wavelets are good for one dimensional jump discontinuities, but are poor in dimensions 2 or more\(^4\).

This is a major weakness, since one of the most widely-lauded applications of wavelet methods is image analysis, which is two-dimensional at its simplest.

In higher dimensions, singularities have a directional character, but wavelets are fundamentally isotropic. This limits wavelets’ effectiveness for resolving key aspects of images, such as edges.

What is needed are decomposition systems that are anisotropic, taking directionality into account.

Shearlets

Starting in the early 2000s, several anisotropic systems were proposed: Curvelets\textsuperscript{5}, Contourlets\textsuperscript{6}, Ridgelets\textsuperscript{7}, Shearlets\textsuperscript{8}, and more.

We note that ridgelets, curvelets, and shearlets fall into the overarching anisotropic paradigm of $\alpha$-molecules.

Let $f \in L^2(\mathbb{R}^2)$ and $\psi$ be a shearlet function. Then $f$ may be decomposed in the following manner, with convergence in the $L^2(\mathbb{R}^2)$ norm:

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^2} \langle f, \psi_{j,k,m} \rangle \psi_{j,k,m}.$$  


Here,

\[ \psi_{j,k,m}(x) := 2^{3j/4} \psi(S_k A_{2^j} x - m). \]

\[ A_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}, \quad S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \]

Note that \( A \) has been replaced with \( A_a \), which is no longer isotropic; this will allow our new analyzing functions to be more pronounced in a particular direction.

The new matrix \( S_k \), a shearing matrix, lets us select the direction.

As \( a \) becomes larger, the direction selected by \( S_k \) will be emphasized to a proportionally greater degree.

Shearlets have the benefit of fast numerical methods, so we focus on them.
One of the theoretical benefits of shearlets is their optimality for representing a certain class of functions.

**Definition**

The set of *cartoon-like images* in $\mathbb{R}^2$ is

$$\mathcal{E} := \{ f \mid f = f_0 + \chi_B f_1, \quad f_i \in C^2([0, 1]^2), \quad \|f_i\|_{C^2} \leq 1, \quad B \subset [0, 1]^2, \quad \partial B \in C^2([0, 1]) \}.$$

The space of cartoon-like images is a quantitative definition of signals that represent images. That is, although images are discrete, if we are to consider only continuous signals, then $\mathcal{E}$ models the class of signals corresponding to images.
Shearlets are known to be near optimal for $\mathcal{E}$ over all possible representation systems\(^9\).

That is, elements of $\mathcal{E}$ may be written with almost optimally few shearlet coefficients, when compared to the number of coefficients required by other representation systems.

From a practical standpoint, this suggest shearlets should be superior to classical methods for the analysis of images.

We shall investigate the efficacy of shearlets for image registration in the final third of this talk.

Goals of this Section

1. Explain the problem of *image registration*.

2. Discuss existing methods for image registration, many of which use harmonic analysis.

3. Detail an algorithm based on *anisotropic features* extracted from the images. This work is joint with Jacqueline Le Moigne (NASA) and David J. Harding (NASA).
The process of image registration seeks to align two or more images of approximately the same scene, acquired at different times or with different sensors.

Images can differ in many ways:

2. Modally: different sensors, different conditions at time of image capture.

Noise may be present.

This problem is relevant to, among other fields, microscopy, biomedical imaging, remote sensing, and image fusion.
- Difficult images to register include those with few dominant features and images from very different sensors, i.e. different modalities.

- We are particularly interested in *multimodal registration*.

- Harmonic analytic techniques are well-suited for these types of problems, when compared to other methods.
Importance of image registration

- Need to be able to know exact location of a newly captured image; this requires registration against a known image.

- Registration is the first step in image fusion.

- Related to more general problems in computer vision.
Any registration algorithm uses the content of the image; how it does so varies substantially.

Difficult images to register include those with few dominant features, and images of different modes.

We are particularly interested in the second case, of *multimodal registration*.

Harmonic analytic techniques are well-suited for these types of problems, when compared to other methods.
A LIDAR and optical image of the Amazon rainforest. These images are very homogeneous. How can we register them? It is hard enough for the human eye to do. Can we use mathematical tools to efficiently extract features to be used for matching?
Image registration may be viewed as the combination of four separate processes:

1. Selecting an appropriate **search space** of admissible transformations. This will depend on whether the images are at the same resolution, and what type of transformations will carry the input image to the reference image, i.e. rotation-scale-translation (RST), polynomial warping, etc.

2. Extracting relevant **features** to be used for matching. These could be individual pixels that are known to be in correspondence between the two images, or could be global structures in the images, such as roads, buildings, rivers, and textures.

3. Selecting a **similarity metric**, in order to decide if a transformed input image closely matches the reference image. This metric should make use of the features which are extracted from the image, be they specific pixels or global structures.

4. Selecting a **search strategy**, which is used to match the images based on maximizing or minimizing the similarity metric.
Manual Registration: A human selects matching pixels in the two images, and the transformation that registers them is computed by minimizing the distance between these pixel pairs.

Algorithmic Pixel Matching: Same as above, but with an algorithm executed automatically. The SIFT algorithm is popular and effective for images of the same modality.

Global Feature Matching: Algorithmically determine robust, sparse features in the images, then compute registration based on these features.

The third class has a strong connection with harmonic analysis.
SIFT fails for multimodal images

Figure: The “matching” pixels computed in the LIDAR and optical images of WA using the SIFT algorithm. Note the lack of correspondence; such points are unusable for a registration algorithm.
In multimodal images, the similarities between the images are only manifested on the *global* scale.

Locally, the images appear dissimilar.

**Figure:** The same alignment of trees in the lidar shaded-relief and optical images of WA. Although there is clear correspondence at the macroscopic level, it is difficult to find pixel-to-pixel correspondences.

We need to use *global features* for these examples.
Wavelet features are well-established for image registration; the separation of edges from textures is often useful for matching. However, the fundamentally isotropic nature of wavelets makes them suboptimal for registering images with strong edge features.

To improve the registration of images with strong edges, we considered features generated from the anisotropic representation system of shearlets. This is good for registration algorithms, because sparse features increase the robustness of the optimization algorithm that computes the registration transformation.

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A Scene from WA

Figure: A grayscale optical image of a mixed land-cover area in Washington state containing both textural and edge-like features. The image is courtesy of Dr. David Harding at NASA GSFC.
Figure: Wavelet (left) and shearlet (right) features extracted from previous image emphasizing textural and edge features, respectively.
When registering images, there are two significant criteria of registration algorithm quality:

1. **Accuracy** of computed registration when compared to the true registration.
2. **Robustness** of algorithm to initial distance between the two images.

The robustness of the algorithm is important because the initial closeness of the two images depends greatly on the GPS technology in the sensors and the distance of the sensing device to the location being imaged.

If the images to be aligned start far apart, the registration algorithm could fail to converge.

We expected using shearlet features would improve robustness.
As we prototyped, we realized that using shearlets did increase robustness, but at a slight cost in accuracy, usually a few pixels. Moreover, shearlets are known the represent edges well, but may not be superior to wavelets for textures.

Consequently, we devised a two-stage registration algorithm: first, use shearlets to get an *approximate registration*, then refine this with another iteration of the algorithm, using wavelet features.

We compared this algorithm to using wavelets alone.

We performed experiments on *synthetic* images, as well as *multimodal* image pairs.
Basic Description of Algorithm

1. **Search Space**: RST. All of our examples feature images at the same scale, so effectively, our search space is the space of rotations and translations (RT).

2. **Features**: Wavelet features in one case and shearlet features coupled with wavelet features in another.

3. **Similarity Metric**: Unconstrained least squares. That is, if $F_R$ and $F_I$ are the reference and input features, $N$ the number of relevant pixels, $(x_i, y_i)$ the integer coordinates of each pixel, and $T_p$ the transformation associated to parameters $p$, we seek to minimize the similarity metric given by

$$\chi^2(p) = \frac{1}{N} \sum_{i=1}^{N} (F_R(T_p(x_i, y_i)) - F_I(x_i, y_i))^2$$

Input a reference image, \( I^r \), and an input image \( I^i \). These will be the images to be registered.

Input an initial registration guess \((\theta_0, T_x^0, T_y^0)\).

Apply shearlet feature algorithm and wavelet feature algorithm to \( I^r \) and \( I^i \). This produces a set of shearlet features for both, denoted \( S^r_1, \ldots, S^r_n \) and \( S^i_1, \ldots, S^i_n \), respectively, as well as a set of wavelet features for both, denoted \( W^r_1, \ldots, W^r_n \) and \( W^i_1, \ldots, W^i_n \). Here, \( n \) refers to the level of decomposition chosen. In general, \( n \) is bounded by the resolution of the images as

\[
n \leq \left\lfloor \frac{1}{2} \log_2(\max\{M, N\}) \right\rfloor,
\]

where \( I^r, I^i \) are \( M \times N \) pixels. For example, a 256 × 256 image would have \( n \leq 4 \).
Match $S'_1$ with $S'_1$ with a least-squares optimization algorithm and initial guess $(\theta_0, T_{x_0}, T_{y_0})$ to get a transformation $T^S_1$. Using $T^S_1$ as an initial guess, match $S'_2$ with $S'_2$, to acquire a transformation $T^S_2$. Iterate this process by matching $S'_j$ with $S'_j$ using $T^S_{j-1}$ as an initial guess, for $j = 2, \ldots, n$. At the end of this iterative matching, we acquire our final shearlet-based registration, call it $T^S = (\theta^S, T^S_x, T^S_y)$.

Using $T^S$ as our initial guess, match $W'_1$ with $W'_1$ with a least-squares optimization algorithm to acquire a transformation $T^W_1$. Using $T^W_1$ as an initial guess, match $W'_2$ with $W'_2$, to acquire a transformation $T^W_2$. Iterate this process by matching $W'_j$ with $W'_j$ using $T^W_{j-1}$ as an initial guess, for $j = 2, \ldots, n$. At the end of this iterative matching, we acquire our final hybrid registration, call it $T^H$.

Output $T^H = (\theta^H, T^H_x, T^H_y)$. 
The wavelet features to be used come in three classes, all implemented in C:

1. Spline wavelets.
2. Simoncelli band-pass wavelet-like features.

The shearlet features are based on the Kaiserslautern MATLAB package, coded by S. Häuser.

All coefficients are thresholded before computing the registration transformation via the Marquadt-Levenberg optimization scheme.
Figure: A synthetic aperture radar image containing several edge-like features.
Figure: Wavelet-like and shearlet features extracted from the original SAR image.
For this talk, we consider seven sets of experiments, in which synthetic images are real multimodal images are registered.

We shall perform many iterations of our algorithm. Each iteration, we shall move the initial guess farther apart.

The distance is parametrized by rotation and translation in the $x$ and $y$ directions. For convenience, these are coupled together as $RT$. So, $RT = 1.8$ means a counterclockwise rotation of 1.8 degrees and a translation of 1.8 pixels in both the $x$ and $y$ direction. Fraction translations and rotations are interpolated by splines.
Figure: In order to produce geometrically warped synthetic input images, we rotated and translated our reference image within the larger source image and extracted the resulting image; the extracted images are indicated by the interior of the black rectangle. The full source image is $1024 \times 1024$, and the extracted images are $256 \times 256$. This extracted input image (bottom) is registered against the extracted reference image (top) in our Mount Hood synthetic experiments. Here, the translation and rotation parameter, $RT$, was set to $RT = 20$. This refers to a counterclockwise rotation of 20 degrees and a translation in the $x$ and $y$ directions by 20 pixels. The images have been converted to grayscale.
Figure: $256 \times 256$ Landsat-7 ETM+ images of Washington D.C. without (left) and with Gaussian noise added (right). The parameters for the noise are mean $\mu = 0$ and variance $\sigma^2 = .05$. The images have been converted to grayscale.
Figure: 512 × 512 lidar shaded relief images of Mossy Rock without (left) and with (right) synthetic radiometric distortion. The images have been converted to grayscale.
Figure: $1024 \times 1024$ images of ETM+ infrared/Red band (left) and near-infrared/NIR band (right) of the Konza Prairie. The images have been converted to grayscale.
Figure: A lidar and aerial photograph for a scene in WA state. Both images are $256 \times 256$. The images have been converted to grayscale.
Figure: Multispectral band 1 (left) and panchromatic band 8 (right) images of Hasselt, Belgium acquired by Landsat ETM+. The images have been converted to grayscale. A subset is extracted from these images to ease computation. The images are courtesy of the IEEE Geoscience and Remote Sensing Society Data Fusion committee.
Figure: Images of MODIS (left) and ETM+ (right) of the Konza Prairie. The MODIS image is $128 \times 128$ and the ETM+ image is $2048 \times 2048$. The images have been converted to grayscale.
<table>
<thead>
<tr>
<th>Experimental Data</th>
<th>Average Improvement over Wavelets-only</th>
<th>Improvement over Best Wavelets-only Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Landsat (synthetic)</td>
<td>237.40%</td>
<td>36.28%</td>
</tr>
<tr>
<td>ETM+ (synthetic)</td>
<td>131.91%</td>
<td>46.27%</td>
</tr>
<tr>
<td>Lidar (synthetic)</td>
<td>87.10%</td>
<td>50.00%</td>
</tr>
<tr>
<td>ETM+ NIR-to ETM+ Red</td>
<td>58.29%</td>
<td>11.76%</td>
</tr>
<tr>
<td>Lidar-to-Optical</td>
<td>5.99%</td>
<td>2.33%</td>
</tr>
<tr>
<td>Multispectral-to-Panchromatic</td>
<td>61.15%</td>
<td>22.73%</td>
</tr>
<tr>
<td>MODIS-to-ETM+</td>
<td>39.68%</td>
<td>39.68%</td>
</tr>
</tbody>
</table>

**Table:** Summary of robustness improvements of hybrid shearlets+wavelets hybrid algorithms over wavelets-only algorithms. We see that the images with strong edge features, such the images in our ETM+ synthetic and MODIS-to-ETM+ experiments, are good images for our hybrid algorithm. Images that are texturally dominant, such as those in our lidar-to-optical experiment, are less appropriate and see less benefit from the hybrid, when compared to wavelets-only.
Experiment conclusions

- For automatic image registration, using shearlets and wavelets together outperforms using only wavelets.

- The improvement is pronounced when there are substantial edges.

- When the image is texturally dominant, there is less noticeable improvement.
Superresolution with Harmonic Analysis
Goals of this Section

1. Introduce the notion of *superresolution*.

2. Briefly survey the mathematical discipline of anisotropic harmonic analysis.

3. Describe a superresolution algorithm based on *shearlets*.

4. Evaluate the results of superresolution experiments on synthetic and hyperspectral data.
The goal of superresolution is to increase the resolution of an image $I$, while preserving detail and without producing artifacts.

The outcome of a superresolution algorithm is an image $\tilde{I}$, which is of the same scene as $I$, but at a higher resolution.

Let $I$ be an $M \times N$ matrix and $\tilde{I}$ an $\tilde{M} \times \tilde{N}$ matrix, with $M < \tilde{M}$, $N < \tilde{N}$. We consider the common case where $\tilde{M} = 2M$ and $\tilde{N} = 2N$, which corresponds to doubling the resolution of the original image.

Images with multiple channels, such as hyperspectral images, can be superresolved by superresolving each channel separately.
Superresolution can be implemented by using information in addition to $I$, such as low resolution images at sub-pixel shifts of the scene or images of the scene with different modalities.

There are several standard approaches to superresolving $I$ without using additional information. Among the most common are nearest neighbor interpolation and bicubic interpolation.

In the case of nearest neighbor interpolation, new pixel values are computed by replicating current pixel values. This method is simple and computationally efficient, but leads to extremely jagged superresolved images.

Other methods involve convolving the image with an interpolation kernel, which amounts to taking a weighted average of pixel values within some neighborhood. For example, bicubic interpolation determines $\tilde{I}$ by computing each new pixel as a weighted average of the 16 nearest neighbors in $I$. 
Harmonic analysis decomposes signals into simpler elements.

Classical methods include Fourier series and wavelets. These have proven extremely influential and quite effective for many applications\textsuperscript{11}.

However, they are fundamentally \textit{isotropic}, meaning they decompose signals without considering how the signal varies \textit{directionally}.

In a broad sense, wavelets decompose an image signal with respect to \textit{translation} and \textit{scale}. Since the early 2000s, there have been several attempts to incorporate directionality into the wavelet construction.

\textsuperscript{11}I. Daubechies. \textit{Ten lectures on wavelets}. Society for industrial and applied mathematics, 1992.
These constructions incorporate directionality in a variety of ways.

Some of the major constructions include:
- Curvelets (Donoho and Candès)\(^2\).
- Contourlets (Do and Vetterli)\(^3\).
- Shearlets (Guo, Labate, Kutyniok, Weiss, et al.)\(^4\).

We consider shearlets for superresolution, since they have several efficient numerical implementations in MATLAB\(^5\).

---


Mathematically, given a signal \( f \in L^2([0, 1]^2) \) and an appropriate shearlet function \( \psi \), we may decompose \( f \) as

\[
f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \langle f, \psi_{i,j,k} \rangle \psi_{i,j,k},
\]

where: \( \psi_{i,j,k}(x) := 2^{3i/4} \psi(B^j A^i x - k) \), \( \begin{pmatrix} 2 & 0 \\ 0 & 2^{1/2} \end{pmatrix} \), \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Note that \( A \) is anisotropic, hence it will allow our new analyzing functions to be more pronounced in a particular direction. The new matrix \( B \), a shearing matrix, lets us select the direction.

The anisotropic character of shearlets has proven useful for a variety of problems in image processing, including image denoising\(^6\) and image registration\(^7\).


Harmonic analysis has been applied to image superresolution, for its ability to capture essential aspects of images.

Particular instances include:

- Wavelets.
- TV-regularization.
- Circulant matrices and frame theory.

The method of circulant matrices is anisotropic; it deduces directional content of an image by adaptively constructing tight frames.

---


The approach taken in our new algorithm is to use anisotropic shearlets to determine the directional content present in an image.

1. An image is up-sampled with a conventional, isotropic superresolution method, for example nearest neighbor or bicubic interpolation.

2. The shearlet transform is then applied to the image, and dominant directions are computed based on these coefficients.

3. This information is then used to smoothly blur the image in certain directions locally, to provide a smoother superresolved image.

Our algorithm for shearlet-based superresolution is coded in MATLAB.
To test our algorithms, we first considered synthetic experiments.

Since our algorithms incorporate anisotropic information, we wanted to study their efficacy on images that have very prominent directional content.

We first constructed $1024 \times 1024$ half planes in MATLAB, at various slopes. Two such half planes appear below.

**Figure:** Half planes of slope $-3$ and $-0.5$, respectively.
We evaluate our algorithm by computing the PSNR for each plane, superresolved with each of four techniques: our shearlets algorithm with 16 and 32 directions, the circulant matrices method of Bosch et al., and bicubic interpolation.

<table>
<thead>
<tr>
<th>Planar Slope</th>
<th>PSNR Shearlet (16 directions)</th>
<th>PSNR Shearlet (32 directions)</th>
<th>PSNR Circulant Matrices</th>
<th>PSNR Bicubic</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>34.3692</td>
<td>34.2751</td>
<td>34.5594</td>
<td>34.2194</td>
</tr>
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<td>0.25</td>
<td>34.3305</td>
<td>34.2185</td>
<td>34.1892</td>
<td>34.2100</td>
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<td>0.33</td>
<td>35.4151</td>
<td>35.4039</td>
<td>35.3457</td>
<td>35.0192</td>
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<td>1.00</td>
<td>35.6006</td>
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<td>2.00</td>
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<td>3.00</td>
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<td>34.5253</td>
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<td>34.2145</td>
</tr>
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<td>4.00</td>
<td>35.6012</td>
<td>34.9848</td>
<td>34.8555</td>
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<td>5.00</td>
<td>35.3447</td>
<td>34.8927</td>
<td>34.6148</td>
<td>34.6299</td>
</tr>
</tbody>
</table>

Table: The PSNR values for various methods on angled half planes.

We notice that the anisotropic harmonic analysis methods outperform bicubic interpolation in all cases.
We also tested our algorithm on real data. We considered a hyperspectral data set of the University of Houston\textsuperscript{11}, which consists of 144 bands of size $349 \times 1905$.

This dataset has a spatial resolution of 2.5 m, and spectral resolution of between 380 nm and 1050 nm. For convenience, we extracted a $256 \times 256$ subset from band 70.

As before, we ran our algorithm for superresolution and compared it to the circulant matrix method and bicubic interpolation. We only considered the shearlet algorithm with 16 directions, due to the size of the image.

<table>
<thead>
<tr>
<th>PSNR Shearlet (16 directions)</th>
<th>PSNR Circulant Matrices</th>
<th>PSNR Bicubic</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.2720</td>
<td>29.2561</td>
<td>29.2125</td>
</tr>
</tbody>
</table>

*Table:* The PSNR values for the University of Houston scene.

The shearlet method produces optimal PSNR for this hyperspectral image.
Figure: Image of Houston, superresolved with bicubic interpolation (left) and shearlet algorithm (right). Notice the smoother edges in the case of shearlet superresolution.
Figure: The direction map computed with shearlet superresolution algorithm.
Figure: Subset from image of Houston, superresolved with bicubic interpolation (left) and shearlet algorithm (right). Notice the smoother edges around circular and rectangular buildings.
We described a shearlet-based algorithm for the superresolution of images.

We performed the algorithm on simple synthetic examples of half planes. We compared our algorithm at two different scales with bicubic interpolation and a recent circulant matrix method.

Our shearlet algorithm tested the best in 5 out of 9 of the half-plane examples, while the circulant matrix method performed the best in the the remaining 4 examples. This indicates that algorithms based on anisotropic harmonic analysis outperform bicubic interpolation in all iterations of our half plane experiments.

We also applied our algorithm to one channel of a hyperspectral image. Compared to bicubic interpolation, our algorithm produced an image having fewer jagged edges and in fact also has improved PSNR.
In general, anisotropic harmonic provides a powerful set of techniques for superresolution, both in terms of PSNR and visual quality.

One of the greatest challenges for our algorithm is superresolving images with many textures without degrading the PSNR.

In future work, we would like to find a method for filtering out the textures so as to only smooth the edges.

In addition, we would like to consider more sophisticated ways of improving edges beyond motion blurring, which tends to decrease image sharpness.

State-of-the-art methods, incorporating directional interpolation and statistical methods, have been proposed by Mallat et al.\textsuperscript{12}, and shall be studied.

It is also of interest to study *data cubes*, such as full hyperspectral or lidar data, with anisotropic harmonic analysis.

These would require 3D transform methods that are more general than the ones described in this talk.

Implementations of 3D shearlets have been recently developed, and are being implemented for the problem of 3D superresolution of hyperspectral and lidar data.