On Fourier Frames

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One of the fundamental results in Fourier analysis is that \( \{e^{i\pi k}\}_{k \in \mathbb{Z}} \) forms an orthogonal basis for \( L^2(-\pi, \pi) \).

An important generalization of an orthogonal basis is a spanning set with possible redundancies in the representation. Such objects are called frames:

**Definition**

A sequence \( \{f_n\}_{n \in \mathbb{N}} \) of elements of \( H \) a Hilbert spaces is a **discrete frame** for \( H \) if:

\[
\exists A, B > 0 \text{ such that } \forall f \in H, \quad A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2.
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Note that we may replace \( \mathbb{N} \) with \( \mathbb{Z} \).
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The first mention of frames was by Duffin and Schaeffer in the context of non-harmonic Fourier series, where families of complex exponentials satisfying the above frame condition were of interest [2].

Their objects of study, so-called *Fourier frames*, shall be the topic of these lectures.

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A \int_{-\pi}^{\pi} |f(x)|^2 \, dx \leq \sum_{k \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} f(x) e^{-i\lambda_k x} \, dx \right|^2 \leq B \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
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Using this notation, $\Lambda = \mathbb{Z}$ generates a Fourier frame, in fact an orthogonal basis.

Central Question: What $\Lambda \subset \mathbb{R}$ generate a Fourier frame?

The main contents of Ortega-Cerdà and Seip’s paper “On Fourier Frames” is a pair of characterizations of such $\Lambda$. Their results will combine a variety of tools from complex analysis, functional analysis and potential theory.
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A key observation regarding the characterization of Fourier frames is that the frame property may, in the context of complex exponentials, be re-cast.

**Definition**

The space of entire functions of exponential type at most $\pi$ whose restriction to $\mathbb{R} \subset \mathbb{C}$ is square-integrable is the *Paley-Wiener space*, denoted $\text{PW}$.

We introduce a new property of sequences $\Lambda \subset \mathbb{R}$:

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In order to relate Fourier frames to sampling sequences, we recall the Paley-Wiener theorem [7]:

**Theorem (Paley-Wiener)**

Let \( \sigma > 0 \) be constant. Then the function \( F(x) \) is of the form

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F(x) = \int_{-\sigma}^{\sigma} f(\xi) e^{i\xi x} \, dx \quad \text{for some } f \in L^2(-\sigma, \sigma)
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if and only if \( F(x) \in L^2(\mathbb{R}) \) and \( F \) can be extended to an entire function of exponential-type at most \( \sigma \), meaning \( F \) extends to an entire function \( \tilde{F} \) such that \( \exists C > 0 \) with the property that \( |\tilde{F}(z)| \leq Ce^{\sigma|z|} \) everywhere.

The Paley-Wiener theorem together with the Plancherel theorem can be used to show that \( \Lambda = \{\lambda_k\}_{k \in \mathbb{Z}} \) is sampling for PW if and only if \( \{e^{i\lambda_k x}\}_{k \in \mathbb{Z}} \) is a Fourier frame. Hence, we will study sampling sequences in order to understand Fourier frames.
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Before discussing the main results of the paper of Ortega-Cerdà and Seip, we present more classical results concerning sampling sequences.

**Definition**

Consider \( \Lambda = \{\lambda_k\}_{k \in \mathbb{Z}} \subset \mathbb{R} \) where \( \lambda_k \leq \lambda_{k+1}, \forall k \in \mathbb{Z} \). Such a sequence is **separated** if \( q := \inf_{k \in \mathbb{Z}} (\lambda_{k+1} - \lambda_k) > 0 \); \( q \) is the **separation constant**. For a separated sequence, define the associated **distribution function** \( n_{\Lambda} \) as follows:

\[
 n_{\Lambda}(0) = 0, \forall a < b, \quad n_{\Lambda}(b) - n_{\Lambda}(a) = |\Lambda \cap (a, b)|.
\]

In particular, a sequence with an accumulation point is not separated. However, some sequences without accumulation points still fail to be separated, for example if we define \( \lambda_k = \sum_{n=1}^{k} \frac{1}{n} \). We shall often assume a sequence to be separated.
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A relatively straightforward inequality related to sampling for PW is:

\[ n_\Lambda(b) - n_\Lambda(a) \geq (1 + \epsilon)(b - a) - C, \ \forall a < b \implies \Lambda \text{ is sampling.} \]

Here \( C, \epsilon \) are of course independent of \( a, b \). The following more sophisticated result gives a necessary condition for sampling, one which involves a logarithmic growth condition on the distribution function.

**Theorem (Landau)**

*If \( \Lambda \) is a separated sampling sequence for PW, then there exist constants \( A, B \), independent of \( a, b \), such that for all \( a < b \):

\[ n_\Lambda(b) - n_\Lambda(a) \geq b - a - A \log^+(b - a) - B. \]

The example of \( \Lambda = \{ k + \log^+ |k| \}_{k \in \mathbb{Z}} \) optimizes Landau’s inequality.*
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Of great use in understanding sampling sequences is the notion of *lower Beurling density*:

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For a separated sequence $\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ with associated distribution $n_\Lambda$, the lower Beurling uniform density is

$$D^-(\Lambda) := \lim_{R \to \infty} \frac{\min_{x \in \mathbb{R}} (n_\Lambda(x + R) - n_\Lambda(x))}{R}.$$ 

Beurling lower density is one way to measure density of a sequence of reals, relative to the integers.
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Beurling lower density is one way to measure density of a sequence of reals, relative to the integers.
Moreover, Beurling lower density gives an almost complete characterization of sampling sequences [3]:

- $D^-(\Lambda) > 1 \implies \Lambda$ is sampling for PW.
- $D^-(\Lambda) < 1 \implies \Lambda$ is not sampling for PW.

So, the critical case is when $D^-(\Lambda) = 1$. This corresponds to a set $\Lambda$ that is “of the same size as the integers,” in the sense of lower Beurling density.
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One of the novelties of the results of Ortega-Cerdà and Seip is their applicability even in the critical case of \( D^-(\Lambda) = 1 \).

We shall now present these two results. We shall present the minimal background required for the results to be intelligible, then proceed to discuss them at length in the second and third lectures.
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**Definition**

The **Hermite-Biehler** space, \( \mathcal{H} \), is the space of entire functions \( f \) without roots in the upper half plane \( \mathbb{H} \) and such that \( |f(z)| \geq |f(\bar{z})| \) whenever \( \Im(z) > 0 \).

A relevant construction involving \( \mathcal{H} \) allows us to construct Hilbert spaces from elements of \( \mathcal{H} \). More explicitly, given \( E \in \mathcal{H} \), we associate a Hilbert space of entire functions:

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H(E) := \left\{ f \text{ entire} \left| \frac{f(z)}{E(z)}, \frac{f^*(z)}{E(z)} \in H^2(\mathbb{H}) \right. \right\}, \quad \|f\|_{H(E)}^2 := \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} \, dt.
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Here, \( f^*(z) := \overline{f(z)} \); this notation shall be used throughout.
Theorem (Main Result 1)

\( \Lambda \subseteq \mathbb{R} \) is sampling for PW if and only if there exist \( E, F \in \overline{HB} \) such that \( H(E) = PW \) and \( \Lambda \) is the zero sequence of \( EF + E^*F^* \).

We shall develop the theory necessary to make sense of this condition in the coming days.
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- We shall develop the theory necessary to make sense of this condition in the coming days.
The second major result concerns generating functions for sampling sequences. More precisely, we shall consider \( \psi \in C^1(\mathbb{R}) \) non-decreasing with the properties that:

1. \( \psi(\infty) - \psi(-\infty) = \infty. \)
2. \( \psi'(x) = o(1) \) as \( |x| \to \infty \) (Recall: \( \psi'(x) = o(1) \) if \( \forall \epsilon > 0, \exists x_0 \) such that \( |\psi'(x)| < \epsilon, \forall x > x_0 \)).

We are interested in the sequence generated by \( \psi \) in the following manner. Consider \( \Lambda(\psi) = \{\lambda_k\}_{k \in \mathbb{Z}} \) given by \( \lambda_k = k - \psi(\lambda_k) \). Alternatively, setting \( \psi(0) = 0 \), this means that \( n_{\Lambda}(t) = \lfloor t + \psi(t) \rfloor \).

**Major Question:** For what \( \psi \) is \( \Lambda(\psi) \) a sampling sequence for PW?
The second major result concerns generating functions for sampling sequences. More precisely, we shall consider $\psi \in C^1(\mathbb{R})$ non-decreasing with the properties that:

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**Major Question:** For what $\psi$ is $\Lambda(\psi)$ a sampling sequence for PW?
A characterization of such $\psi$ is given in terms of the extent to which the potential of $\psi$ can be approximated by elements of $\overline{HB}$.

**Definition**

For a $\psi$ with the above properties, the corresponding potential is given by:

$$U_\psi(z) := \int_{-\infty}^{\infty} \left[ \log \left| 1 - \frac{z}{t} \right| + \Re \left( \frac{z}{t} \right) \right] d\psi(t),$$

taken in the principle value sense.

A crucial property of $U_\psi$ is that it is sub-harmonic.
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A crucial property of $U_\psi$ is that it is sub-harmonic.
Our characterization will involve a notion of *logarithmically regular partition*.

**Definition**

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be such that $t_0 = 0$ and $\psi(t_n) = n$, $\forall n \geq 1$. Set $d_n := t_n - t_{n-1}$. We say $\psi$ induces a *logarithmically regular partition* if $d_n \sim d_{n+1}$ and

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\sup_{x>0} \sum_{\frac{x}{2} < t_n < 2x} \frac{d_n^2}{(x - t_n)^2 + d_n^2} = \sup_{x>0} \sum_{\frac{x}{2} < t_n < 2x} \frac{(t_n - t_{n-1})^2}{(x - t_n)^2 + (t_n - t_{n-1})^2} < \infty.
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Theorem (Main Result 2)

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. If $\psi'(x) = \frac{1}{O(x)}$ when $x \to \infty$ and $\psi$ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.

2. If $\psi'(x) = o\left(\frac{1}{x}\right)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

Note that $\psi'(x) = \frac{1}{O(x)}$ if $\exists M > 0$, $x_0$ such that $\left|\frac{1}{Mx}\right| \leq |\psi'(x)|$, $\forall x > x_0$.

$\psi'(x) = o\left(\frac{1}{x}\right)$ if $\forall \epsilon > 0$, $\exists x_0$ such that $|\psi'(x)| < \left|\frac{\epsilon}{x}\right|$, $\forall x > x_0$. 
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Having presented the two fundamental results of Ortega-Cerdà and Seip’s “On Fourier Frames” in relative isolation, we introduce the machinery deployed to prove them.

Main result 1 will be proven using a series of lemmas and theorems from complex and functional analysis, while main result 2 will be somewhat more self-contained. The proof of the latter will however draw heavily from ideas of Lyubarskii and Malinnikova [4] and their work on approximating subharmonic functions.
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We begin by introducing de Branges’ theory of Hilbert spaces of entire functions.

**Definition**

A *de Branges space* is a Hilbert space $H$ of entire functions with the following three properties:

1. If $f \in H$, $\zeta$ non-real such that $f(\zeta) = 0$, then $g \in H$, where $g(z) := \frac{f(x)(z - \overline{\zeta})}{z - \zeta}$. Moreover, $\|f\|_H = \|g\|_H$.

2. For every $\zeta$ non-real, the linear functional on $H$ given by $\zeta \mapsto f(\zeta)$ is continuous.

3. If $f \in H$, then $f^* \in H$, where $f^*(z) := \overline{f(\overline{z})}$. 
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Recall the Hermite-Biehler space of entire functions:

**Definition**

The *Hermite-Biehler* space, $\overline{HB}$, is the space of entire functions $f$ without roots in the upper half plane $\mathbb{H}$ and such that $|f(z)| \geq |f(\bar{z})| = |f^*(z)|$ whenever $\Im(z) > 0$.

Note that by the maximum modulus principle, the above condition may be replaced by $|f(z)| > |f(\bar{z})|$ when $\Im(z) > 0$.

It is not difficult to show that $\overline{HB}$ is a de Branges space. What is of far greater interest is the following characterization of de Branges spaces.
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Recall our construction for $H(E)$: given $E \in \overline{HB}$, we associate a Hilbert space of entire functions:

$$H(E) := \left\{ f \text{ entire} \, \middle| \, \frac{f(z)}{E(z)}, \frac{f^*(z)}{E(z)} \in H^2(\mathbb{H}) \right\}, \quad \|f\|_{H(E)}^2 := \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} dt.$$ 

**Theorem (Characterization of de Branges spaces)**

A Hilbert space of entire functions satisfying the three criterion of a de Branges space is equal isometrically to $H(E)$, some $E \in \overline{HB}$.

In other words, up to isometry, the $H(E)$ are exactly the de Branges spaces.
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In other words, up to isometry, the $H(E)$ are exactly the de Branges spaces.
The second condition of a de Branges space $H$, namely that for every non-real $\zeta$, the linear functional on $H$ given by $f \mapsto f(\zeta)$ is continuous, has the consequence that each such $\zeta$ yields a reproducing kernel $K_E(\zeta, z)$:

**Theorem (Reproducing Kernel for $H(E)$)**

Let $E \in \overline{HB}$. For each fixed $\zeta \in \mathbb{C}$, the function

$$K_E(\zeta, z) := \frac{i}{2} \frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{\pi(z - \overline{\zeta})}$$

as a function of $z$ is in $H(E)$. Moreover, $K_E$ is a reproducing kernel for $H(E)$:

$$\forall f \in H(E), \quad \langle f, K_E(\zeta, \cdot) \rangle_E = \int_{-\infty}^{\infty} \frac{f(t)\overline{K_E(\zeta, t)}}{|E(t)|^2} dt = f(\zeta).$$
Useful in analyzing $E \in \overline{HB}$ will be a notion of *phase function*.

**Proposition**

For $x \in \mathbb{R}$, we may write $E(x) = |E(x)|e^{-i\phi(x)}$, where $\phi(x) \in C(\mathbb{R})$ is such that $E(x)e^{i\phi(x)} \in \mathbb{R}$, for all $x \in \mathbb{R}$.

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If $x \neq 0$, then a direct computation yields:

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\|K_E(x, \cdot)\|^2_E = K_E(x, x) = \frac{1}{\pi} \phi'(x)|E(x)|^2.
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This useful identity allows to us to prove, among other things, the following Plancherel-type result for de Branges space:
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Theorem (Generalized Plancherel)

Let $H(E)$ be a de Branges space, $\phi$ the phase function associated to $E$. Suppose $\alpha \in \mathbb{R}$ and let $\Gamma := \{\gamma_k\}_{k \in \mathbb{Z}}$ be the sequence of real numbers such that $\phi(\gamma_k) = \alpha + k\pi$, $k \in \mathbb{Z}$. Then if $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, the family of normalized reproducing kernels

$$\left\{ \frac{K_E(\gamma_k, z)}{\|K_E(\gamma_k, \cdot)\|_E} \right\}_{k \in \mathbb{Z}}$$

is an orthonormal basis for $H(E)$. In particular:

$$\|f\|_E^2 = \sum_k \frac{\pi |f(\gamma_k)|^2}{\phi'(\gamma_k)|E(\gamma_k)|^2}, \forall f \in H(E).$$

Note that $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$ for at most one $\alpha \in [0, \pi)$, so the conditions of the theorem are easily met.
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One more crucial theorem of the de Branges theory will be needed to prove main result 1. It relates the classical Poisson transform to the norm on $H(E)$.

**Theorem**

Let $\mu$ be a measure on $\mathbb{R}$, and $E \in \overline{HB}$. Then:

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\int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} \, d\mu(t) = \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} \, dt
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if and only if there exists a bounded holomorphic function $A$ on $\mathbb{H}$ such that

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We are now ready to sketch a proof of main result 1.

It has been shown by Seip [6] that any $\Lambda$, a sampling sequence for PW, contains $\Lambda' \subseteq \Lambda$ that is sampling and separated. Thus, we may WLOG restrict ourselves to separated $\Lambda$.

**Theorem (Main Result 1)**

$\Lambda \subset \mathbb{R}$ is a separated sampling sequence for PW if and only if there exist $E, F \in \text{HB}$ such that $H(E) = PW$ and $\Lambda$ is the zero sequence of $EF + E^*F^*$. 
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We first prove the $\Rightarrow$ implication, so assume $\Lambda \subset \mathbb{R}$ is a separated sampling sequence for $PW$.

$PW$ with norm $f \mapsto \sqrt{\sum_k |f(\lambda_k)|^2}$ is a de Branges space. So by our characterization of de Branges spaces, there exists $E \in \overline{HB}$ such that $H(E) = PW$ and $\sum_k |f(\lambda_k)|^2 = \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt$.

Setting $\mu = \sum_k |E(\lambda_k)|^2 \delta_k$ and applying our theorem on the Poisson transform, we get a bounded holomorphic function $A$ in $H$ such that $\|A\|_{\infty} \leq 1$, and $a \in \mathbb{R}$ such that:

$$-i \sum_k |E(\lambda_k)|^2 \left( \frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right) + ia = \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)}$$
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We first prove the \( \Rightarrow \) implication, so assume \( \Lambda \subset \mathbb{R} \) is a separated sampling sequence for \( PW \).

\[ PW \text{ with norm } f \mapsto \sqrt{\sum_k |f(\lambda_k)|^2} \text{ is a de Branges space. So by our characterization of de Branges spaces, there exists } E \in \overline{HB} \text{ such that } H(E) = PW \text{ and } \sum_k |f(\lambda_k)|^2 = \int \frac{|f(t)|^2}{|E(t)|^2} dt. \]

Setting \( \mu = \sum_k |E(\lambda_k)|^2 \delta_k \) and applying our theorem on the Poisson transform, we get a bounded holomorphic function \( A \) in \( \mathbb{H} \) such that \( \|A\|_\infty \leq 1 \), and \( a \in \mathbb{R} \) such that:

\[ -i \sum_k |E(\lambda_k)|^2 \left( \frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right) + ia = \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)} \]
We now analyze this equation. Notice the LHS is meromorphic, but the RHS is holomorphic in $\mathbb{H}$. Set

$$M(z) := -i \sum_k |E(\lambda_k)|^2 \left( \frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right) + ia$$

It is readily verified that

$$A = \frac{M - 1}{M + 1} \frac{E}{E^*}.$$

Notice $M - 1$ has poles exactly at the points $\lambda_k$. Moreover, our main equality tells us $M - 1$ vanishes whenever $E^*$ does.

We may thus write:

$$M - 1 = -\frac{E^*F^*}{G}, \quad F \text{ entire, } G(z) := \prod_k \left( 1 - \frac{z}{\lambda_k} \right) e^{\frac{z}{\lambda_k}}.$$

It is readily verified that $M^* = -M$ and $G^* = G$. 
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It is readily verified that \( M^* = -M \) and \( G^* = G \).
We conclude $M + 1 = \frac{EF}{G}$.

This implies $\frac{F^*}{F} = -A$ in $H$ and $F$ has no zeroes in $H$. Since $\|A\|_\infty \leq 1$, we conclude $F \in HB$.

We now claim that $G = \frac{EF + E^*F^*}{2}$, which will imply $\Lambda$ is the zero sequence of $EF + E^*F^*$, since $\Lambda$ is by definition the zero sequence for $G$. 

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Now, we know:

\[-MG + EF\]

\[= - \left( \frac{EF}{G} - 1 \right) G + EF\]

\[= G\]

If $x \in \mathbb{R}$, then by construction $G(x)$ is real and $M(x)G(x)$ is imaginary. So by elementary complex analytic techniques, $G = \Re(EF)$.

We conclude $G(z) = \frac{EF + E^*F^*}{2}$, $\forall z \in \mathbb{C}$, as desired.
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We now turn to the converse. Assume we have \( E, F \in \overline{HB} \) such that \( PW = H(E) \) and \( \Lambda \) is the zero sequence of \( EF + E^*F^* \). We shall prove \( \Lambda \) is sampling for \( PW \).

Notice \( H(E) = PW \) implies \( E \) has no real zeroes. WLOG, \( F \) also has no real zeroes.

For \( \alpha \in (0, \pi] \), we define \( \Lambda_\alpha = \{ \lambda_{\alpha,k} \}_{k \in \mathbb{Z}} \) by \( \phi_{EF}(\lambda_{\alpha,k}) = \alpha + k\pi \).

Observe that since \( \Lambda \) is the zero sequence of \( EF + E^*F^* \), and \( E, F \) have no real zeroes, \( \Lambda = \Lambda_{\pi/2} \).
We now turn to the converse. Assume we have $E, F \in \overline{HB}$ such that $PW = H(E)$ and $\Lambda$ is the zero sequence of $EF + E^*F^*$. We shall prove $\Lambda$ is sampling for $PW$.

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Observe that since $\Lambda$ is the zero sequence of $EF + E^*F^*$, and $E, F$ have no real zeroes, $\Lambda = \Lambda_{\frac{\pi}{2}}$. 
For $\alpha \neq \frac{\pi}{2}$, $\Lambda_\alpha$ is interlaced with $\Lambda$. Since $\Lambda$ is separated, $\Lambda_\alpha$ can be expressed as the union of two separated sequences, each with separation constant greater than or equal to that of $\Lambda$.

Citing the Plancherel-Pólya inequality, we conclude that there exists $C$, independent of $\alpha$, such that:

$$\sum_{k} |f(\lambda_{\alpha,k})|^2 \leq C \|f\|^2_{PW}.$$

Now, applying our generalized Plancherel theorem, we have that for all but at most one $\alpha \in (0, \pi]$:

$$\forall g \in H(EF), \int_{\mathbb{R}} \frac{|g(t)|^2}{|E(t)F(t)|^2} dt = \sum_{k} \frac{|g(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})F(\lambda_{\alpha,k})|^2 \phi'_{EF}(\lambda_{\alpha,k})}.$$
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It is clear that for every $f \in H(E)$, the function $g := fF \in H(EF)$, and $\|f\|_{H(E)} = \|fF\|_{H(EF)}$.

Consequently, for every $f \in PW$, we have:

$$\|f\|_{PW}^2 \sim \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt = \int_{\mathbb{R}} \frac{|g(t)|^2}{|E(t)F(t)|^2} dt = \sum_{k} \frac{|f(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})|^2 \phi_{EF}^\prime(\lambda_{\alpha,k})}.$$
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Consequently, for every $f \in PW$, we have:

$$\|f\|_{PW}^2 \approx \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt = \int_{\mathbb{R}} \frac{|g(t)|^2}{|E(t)F(t)|^2} dt = \sum_k \frac{|f(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})|^2 \phi'_{EF}(\lambda_{\alpha,k})}.$$
Recalling $E$ has no real zeroes, we see for $x \in \mathbb{R}$:

$$1 = \sup_{f \in PW, \|f\|_{PW}^2 \leq 1} |f(x)|^2 \simeq \sup_{f \in H(E), \|f\|_{H(E)}^2 \leq 1} |f(x)|^2 = K_E(x, x) = \frac{1}{\pi} \phi'_E(x)|E(x)|^2$$

Since $\phi'_{EF} = \phi'_E + \phi'_F \geq \phi'_E$, the above inequality along with the previous string of equalities yields:

$$\|f\|_{PW}^2 \leq C \sum_k |f(\lambda_{\alpha,k})|^2$$

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We must note that this inequality could fail for a single $\alpha \in [0, \pi)$, namely the $\alpha$ for which generalized Plancherel could fail. WLOG, $\alpha = \frac{\pi}{2}$, for otherwise we have already established that $\Lambda$ is sampling.

If $\alpha = \frac{\pi}{2}$, then let $\{\alpha_n\} \to \frac{\pi}{2}$. Notice that $\sum_k |f(\lambda_{\alpha_n,k})|^2 \leq C\|f\|_{PW}^2$.

Applying the Lebesgue Dominated Convergence Theorem, we achieve the desired inequality for $\alpha = \frac{\pi}{2}$ as well.

This shows $\Lambda$ is sampling, so main result 1 is proven.
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Applying the Lebesgue Dominated Convergence Theorem, we achieve the desired inequality for $\alpha = \frac{\pi}{2}$ as well.

This shows $\Lambda$ is sampling, so main result 1 is proven.
We now present an interpretation of the function $F$ in main result 1.

**Definition**

\[ \Lambda = \{ \lambda_k \}_{k \in \mathbb{Z}} \]  is a complete interpolating sequence if the interpolation problem

\[ f(\lambda_k) = a_k, \quad k \in \mathbb{Z}, \]  has a unique solution \( f \in PW \) for all \( l^2 \) data \( \{ a_k \}_{k \in \mathbb{Z}} \).

We note that an alternate characterization of \( \Lambda \) being a complete interpolating sequence is that \( \Lambda \) is sampling, but \( \Lambda \setminus \{ \lambda_j \} \) is not, for any \( j \in \mathbb{Z} \).
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**Definition**

$\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ is a *complete interpolating sequence* if the interpolation problem $f(\lambda_k) = a_k, k \in \mathbb{Z}$, has a unique solution $f \in PW$ for all $l^2$ data $\{a_k\}_{k \in \mathbb{Z}}$.

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If \( \Lambda \) is a complete interpolating sequence, then our characterization of de Branges spaces and our generalized Plancherel theorem imply \( \exists E \in \overline{HB} \) such that \( H(E) = PW \) and \( \Lambda \) constitutes the zero sequence of \( E + E^* \).

In this sense, we may understand \( F \) as accounting for the redundancy in \( \Lambda \).

In particular, if \( D^-(\Lambda) > 1 \), Seip has shown \( \Lambda = \Lambda' \cup (\Lambda \setminus \Lambda') \), where \( \Lambda' \) is a complete interpolating sequence. In this case, the hypotheses of main result 1 are met if we choose \( E \) to correspond as above with \( \Lambda' \) and set \( F \) to be:

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F(z) := \prod_{\lambda_k \in \Lambda \setminus \Lambda'} \left(1 - \frac{z}{\lambda_k}\right) e^{\frac{z}{\lambda_k}}.
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If $\Lambda$ is a complete interpolating sequence, then our characterization of de Branges spaces and our generalized Plancherel theorem imply $\exists E \in \overline{HB}$ such that $H(E) = PW$ and $\Lambda$ constitutes the zero sequence of $E + E^*$. 

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Another interpretation of $F$ is formulated by extending the notion of complete interpolating sequences to de Branges spaces:

**Definition**

Let $H$ be a de Branges space. $\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ is a complete interpolating sequence for a de Branges space if the interpolation problem $f(\lambda_k) = a_k$, $k \in \mathbb{Z}$ has a unique solution $f \in H$ for all $\{a_k\}_{k \in \mathbb{Z}}$ such that

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Our main result gives that $\Lambda$ sampling for $PW$ implies $\Lambda$ is a complete interpolating sequence for $H(EF)$ and $H(E) = PW$ is isometrically embedded into $H(EF)$ by the map $f \mapsto fF$.

This relates to a general result of Seip [6], which states that we cannot in general take a sampling sequence $\Lambda$ and acquire a complete interpolating sequence as a subsequence, that is to say by making $\Lambda$ thinner. Instead, we can make the space larger so that $\Lambda$ becomes a complete interpolating sequence for the larger space.
Our main result gives that \( \Lambda \) sampling for \( PW \) implies \( \Lambda \) is a complete interpolating sequence for \( H(EF) \) and \( H(E) = PW \) is isometrically embedded into \( H(EF) \) by the map \( f \mapsto f_F \).

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An interesting corollary of main result 1 relates separated sampling sequences and complete interpolating sequences:

**Corollary**

If $\Lambda$ is a separated sampling sequence for $PW$, there exists a complete interpolating sequence $\Gamma = \{\gamma\}_{k \in \mathbb{Z}}$ such that for every $k \in \mathbb{Z}$, there is at least one $\lambda \in \Lambda$ such that $\gamma_k \leq \lambda \leq \gamma_{k+1}$.

**Proof.**

We have mentioned that $\Lambda$ sampling implies $\Lambda$ consists of those points $\{\lambda\}$ such that $\phi_{EF}(\lambda) = \frac{\pi}{2} + k\pi$, for some $k \in \mathbb{Z}$. On the other hand, $\phi_E$ is increasing and increases more slowly than $\phi_{EF}$. The set $\Gamma_\alpha = \{\gamma \mid \phi_E(\gamma) = \alpha + k\pi\}$ thus has the desired property. Moreover, since $H(E) = PW$, generalized Plancherel implies $\Gamma_\alpha$ is a complete interpolating sequence except for at most one $\alpha \in [0, \pi)$. Pick a good one.
An interesting corollary of main result 1 relates separated sampling sequences and complete interpolating sequences:

**Corollary**

If $\Lambda$ is a separated sampling sequence for $\text{PW}$, there exists a complete interpolating sequence $\Gamma = \{\gamma\}_{k \in \mathbb{Z}}$ such that for every $k \in \mathbb{Z}$, there is at least one $\lambda \in \Lambda$ such that $\gamma_k \leq \lambda \leq \gamma_{k+1}$.

**Proof.**

We have mentioned that $\Lambda$ sampling implies $\Lambda$ consists of those points $\{\lambda\}$ such that $\phi_{EF}(\lambda) = \frac{\pi}{2} + k\pi$, for some $k \in \mathbb{Z}$. On the other hand, $\phi_E$ is increasing and increases more slowly than $\phi_{EF}$. The set $\Gamma_\alpha = \{\gamma \mid \phi_E(\gamma) = \alpha + k\pi\}$ thus has the desired property. Moreover, since $H(E) = \text{PW}$, generalized Plancherel implies $\Gamma_\alpha$ is a complete interpolating sequence except for at most one $\alpha \in [0, \pi)$. Pick a good one.
An interesting corollary of main result 1 relates separated sampling sequences and complete interpolating sequences:

**Corollary**

If $\Lambda$ is a separated sampling sequence for $PW$, there exists a complete interpolating sequence $\Gamma = \{\gamma\}_{k \in \mathbb{Z}}$ such that for every $k \in \mathbb{Z}$, there is at least one $\lambda \in \Lambda$ such that $\gamma_k \leq \lambda \leq \gamma_{k+1}$.

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We now move to the case when $D^-(\Lambda) = 1$ but there does not exist $\Lambda' \subset \Lambda$ such that $\Lambda'$ is a complete interpolating sequence.

Let $\psi \in C^1(\mathbb{R})$ be non-decreasing such that $\psi(\infty) - \psi(-\infty) = \infty$ and $\psi'(x) = o(1)$ as $|x| \to \infty$.

To $\psi$ we associate a sequence $\Lambda(\psi) = \{\lambda_k\}_{k \in \mathbb{Z}}$ given by $\lambda_k = k - \psi(\lambda_k)$.

Alternatively, setting $\psi(0) = 0$, $n_{\Lambda(\psi)}(t) = \lceil t + \psi(t) \rceil$.
We now move to the case when \( D^- (\Lambda) = 1 \) but there does not exist \( \Lambda' \subset \Lambda \) such that \( \Lambda' \) is a complete interpolating sequence.

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Alternatively, setting \( \psi(0) = 0 \), \( n_{\Lambda(\psi)}(t) = [t + \psi(t)] \).
All such $\Lambda(\psi)$ are *sets of uniqueness*, i.e. every trigonometric series vanishing off of $\Lambda(\psi)$ is identically zero.

However, Seip [6] has shown that no $\Lambda(\psi)$ can contain a complete interpolating sequence as a subset.

In order to understand $\Lambda(\psi)$ in the context of sampling, we introduce the following potential function:

$$U_{\psi}(z) := \int_{\mathbb{R}} \left( \log \left| 1 - \frac{z}{t} \right| + \Re \left( \frac{z}{t} \right) \right) d\psi(t).$$

Note that $\psi' \geq 0$ implies $U_{\psi}$ is sub-harmonic.
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We recall the second main result of the paper of Oretega-Cerdà and Seip:

**Definition**

Let \( \psi \) be as above, and WLOG assume that \( \psi(x) \equiv 0 \) for \( x \leq 0 \). Let \( \{t_n\}_{n=0}^\infty \) be such that \( t_0 = 0 \) and \( \psi(t_n) = n, \ \forall n \geq 1 \). Set \( d_n := t_n - t_{n-1} \). We say \( \psi \) induces a logarithmically regular partition if \( d_n \simeq d_{n+1} \) and

\[
\sup_{x>0} \sum_{\frac{x}{2} < t_n < 2x} \frac{d_n^2}{(x - t_n)^2 + d_n^2} < \infty.
\]

**Theorem (Main Result 2)**

Let \( \psi \) be as above, and WLOG assume that \( \psi(x) \equiv 0 \) for \( x \leq 0 \). Then:

1. **If** \( \psi'(x) = \frac{1}{O(x)} \) when \( x \to \infty \) and \( \psi \) induces a logarithmically regular partition, then \( \Lambda(\psi) \) is sampling for PW.
2. **If** \( \psi'(x) = o\left(\frac{1}{x}\right) \) when \( x \to \infty \), then \( \Lambda(\psi) \) is not sampling for PW.
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$$\sup_{x>0} \sum_{\frac{x}{2} < t_n < 2x} \frac{d_n^2}{(x - t_n)^2 + d_n^2} < \infty.$$ 

**Theorem (Main Result 2)**

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. If $\psi'(x) = O\left(\frac{1}{x}\right)$ when $x \to \infty$ and $\psi$ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.
2. If $\psi'(x) = o\left(\frac{1}{x}\right)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.
In order to prove main result 2, we first establish the relationship between sampling for PW and the extent to which $U_\psi$ can be approximated by the logarithm of the modulus of an entire function. This will follow in part from main result 1.

**Corollary**

$\Lambda(\psi)$ is sampling for PW if there exists $f \in \overline{HB}$ such that $\phi'_f(x) = o(1)$ when $|x| \to \infty$ and such that:

$$|U_\psi(z) - \log |f(z)|| \lesssim 1, \Im(z) \geq 0.$$  

Notice if we could find $e \in \overline{HB}$ such that $\phi_e(x) = \pi x + \pi \psi(x) - \phi_f(x)$, we would be done. This is because $\Lambda(\psi)$ would be the zero sequence of $ef + e^*f^*$ and $|e(z)| = e^{\pi \Im(z)}$ for $\Im(z) \geq 0 \implies H(e) = PW$. We could thus apply main result 1 and conclude $\Lambda(\psi)$ is sampling.
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Sadly, such an $e$ is elusive. Instead, we shall prove the result using the following perturbation principle, proved originally by Duffin and Schaeffer [2]:

**Proposition (Perturbation Principle)**

If $\Gamma = \{\gamma_k\}$ is sampling, then $\Gamma' = \{\gamma_k + \delta_k\}$ is sampling whenever each $\gamma_k + \delta_k$ is distinct and $\delta_k \to 0$ as $|k| \to \infty$.

So, it suffices to find $E \in \overline{HB}$ such that $\phi_E(x) - \pi x - \pi \psi(x) + \phi_f(x) = o(1)$ as $|x| \to \infty$ and $|E(x)| \simeq e^{\pi \Im(z)}$ for $\Im(z) \geq 0$.

This is because we can then apply the perturbation principle with the zero sequence of $Ef + E^*f^*$ playing the role of $\Gamma$ and $\Lambda(\psi)$ playing the role of $\Gamma'$. 
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\]
as \( |x| \to \infty \) and \( |E(x)| \approx e^{\pi \Im(z)} \) for \( \Im(z) \geq 0 \).

This is because we can then apply the perturbation principle with the zero sequence of \( Ef + E^*f^* \) playing the role of \( \Gamma \) and \( \Lambda(\psi) \) playing the role of \( \Gamma' \).
• WLOG, $\omega(x) := x + \psi(x) + \frac{1}{\pi} \phi_f(x)$ satisfies $1 + \psi'(x) + \frac{1}{\pi} \phi'_f = \omega' \simeq 1$.

• Partition $\mathbb{R}$ into a sequence of disjoint intervals $I_k = [x_k, x_{k+1}], k \in \mathbb{Z}, x_0 = 0$, such that:

$$\int_{I_k} \omega'(t)dt = x_{k+1} - x_k + \psi(x_{k+1}) - \psi(x_k) + \frac{1}{\pi} \phi_f(x_{k+1}) - \frac{1}{\pi} \phi_f(x_k) = 1, \ \forall k.$$ 

• Now, choose $\gamma_k \in I_k$ such that $\gamma_k = \int_{I_k} t\omega'(t)dt$. 
WLOG, \( \omega(x) := x + \psi(x) + \frac{1}{\pi} \phi_f(x) \) satisfies \( 1 + \psi'(x) + \frac{1}{\pi} \phi'_f = \omega' \approx 1 \).

Partition \( \mathbb{R} \) into a sequence of disjoint intervals \( l_k = [x_k, x_{k+1}] \), \( k \in \mathbb{Z} \), \( x_0 = 0 \), such that:

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Now, choose $\gamma_k \in I_k$ such that $\gamma_k = \int_{I_k} t \omega'(t) dt$. 

Set \( A(z) := \lim_{R \to \infty} \prod_{|\gamma_k| < R} \left( 1 - \frac{z}{\gamma_k} \right) \) and \( \Gamma = \{\gamma_k\} \).

Ortega-Cerdá and Seip have previously shown that:

\[
|A(z)| e^{-U_\omega(z)} \simeq \min(1, \text{dist}(z, \Gamma)).
\]

Now choose two monomials \( P, Q \) of the same degree with only real zeroes such that the function \( B \) defined by:

\[
B(z) := \frac{A(z - \frac{1}{2})P(z)}{Q(z)}
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By a theorem of Meiman, either \( A - iB \in \overline{HB} \) or \( A + iB \in \overline{HB} \); we set \( E = A - iB \) and WLOG (since \( P \) may be replaced by \( -P \)) we may assume \( E \in \overline{HB} \).

It remains to show \( E \) satisfies our desired properties, namely, that 
\[
\phi_E(x) - \pi x - \pi \psi(x) + \phi_f(x) = o(1) \quad \text{and} \quad |E(z)| \sim e^{\pi \Im(z)}, \quad \Im(z) \geq 0.
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The fact that 
\[
|A(z)| e^{-U_\omega(z)} \sim 1
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together with the hypothesis
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give that \( \phi_E(x) - \pi x - \pi \psi(x) + \phi_f(x) = o(1) \).
Consider the sequence of functions $A_k(z) := A(z - x_{2k})$. A normal family argument shows $\exists \{c_k\} \simeq 1$ such that $A_k(x) - c_k \cos(\pi x) \to 0$ uniformly on compact subsets of $\mathbb{R}$.

Similarly, we may set $B_k(x) := B(z - x_{2k})$ and obtain that $B_k(x) - c_k \sin(\pi x) \to 0$ uniformly on compact subsets of $\mathbb{R}$. This gives $|E(z)| \simeq e^{\pi \Im(z)}$, $\Im(z) \geq 0$ and the result follows.
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We now turn to the second main result of the paper of Ortega-Cerdá and Seip:

**Theorem (Main Result 2)**

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. If $\psi'(x) = \frac{1}{o(x)}$ when $x \to \infty$ and $\psi$ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.

2. If $\psi'(x) = o\left(\frac{1}{x}\right)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

**Remark:** $\psi(x) := \sqrt{x} \chi_{[0,\infty)}$ corresponds to a sampling sequence with no subsequence a complete interpolating sequence; this is due to Seip.
We now turn to the second main result of the paper of Ortega-Cerdá and Seip:

**Theorem (Main Result 2)**

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. If $\psi'(x) = O\left(\frac{1}{\sigma(x)}\right)$ when $x \to \infty$ and $\psi$ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.

2. If $\psi'(x) = o\left(\frac{1}{x}\right)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

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We now turn to the second main result of the paper of Ortega-Cerdá and Seip:

**Theorem (Main Result 2)**

Let $\psi$ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. If $\psi'(x) = \frac{1}{\Theta(x)}$ when $x \to \infty$ and $\psi$ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.
2. If $\psi'(x) = o\left(\frac{1}{x}\right)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

**Remark:** $\psi(x) := \sqrt{x} \chi_{[0, \infty)}$ corresponds to a sampling sequence with no subsequence a complete interpolating sequence; this is due to Seip.
We shall prove the first claim of this theorem via our corollary, and the second claim directly. The key for the second claim is to show $\psi'(x) = \frac{1}{O(x)}$ is critical.

We begin with the first claim, namely that $\psi'(x) = \frac{1}{O(x)}$ as $x \to \infty$ and $\psi$ inducing a logarithmically regular partition together imply $\Lambda(\psi)$ is sampling for PW.

By our corollary, it suffices to produce an appropriate $f \in \overline{HB}$. We define the zeroes of $f$ as follows: let $\{t_n\}$ be the logarithmically regular partition, and define $r_n \in (t_{n-1}, t_n)$ by:

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Now, set $z_n := r_n e^{-\frac{i cd_n}{r_n}}$, where $c > 0$ is small enough that $\frac{cd_n}{r_n} \leq \frac{\pi}{4}$, $\forall n$. 
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Now, choose $f$ so that:

$$\log |f(z)| = \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \left( \log \left| 1 - \frac{Z}{Z_n} \right| + \Re \left( \frac{Z}{t} \right) \right) d\psi(t).$$

We see $f \in \overline{HB}$. Set $V := U_\psi - \log |f|$; this satisfies:

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It suffices to prove this series converges uniformly on compact subsets of $\mathbb{C}$, and that $V(z) = O(1)$ for $\Im(z) \geq 0$, since we then apply our corollary and conclude $\Lambda(\psi)$ is sampling for PW.
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It suffices to prove this series converges uniformly on compact subsets of $\mathbb{C}$, and that $V(z) = O(1)$ for $\Im(z) \geq 0$, since we then apply our corollary and conclude $\Lambda(\psi)$ is sampling for PW.
For a given $z \in \mathbb{C}$, let $n = n(z)$ be a positive integer such that $r_{n-1} < |z| \leq r_n$.

If $\Im(z) \geq 0$, the smoothness of $\psi$ ensures that $n(z) + 1 \sum_{k=n(z)-1}^{n(z)+1} j_n(z) \approx 1$.

Next, let $n^-(z) \in \mathbb{Z}^+$ be such that $r_{n^-(z)-1} < \frac{|z|}{2} \leq r_{n^-(z)}$ and $n^+(z) \in \mathbb{Z}^+$ be such that $r_{n^+(z)-1} < 2|z| \leq r_{n^+(z)}$; these are necessarily unique.

For $z \in \mathbb{C}$, we see that $\sum_{k=n^+(z)+1}^{\infty} j_k(z) \approx 1$; indeed following an argument in [4], there is a constant $C_1 > 0$ such that each term in this summation satisfies $|j_n(z)| \lesssim C_1^{-(n-n^+(z)-1)}$. 
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We now re-write:

\[ j_n(z) = \int_{r_{n-1}}^{r_n} \left( \log \left| 1 - \frac{Z}{t} \right| - \log \left| 1 - \frac{Z}{Z_n} \right| \right) d\psi(t) \]

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It remains to show $V(z) \simeq 1$. To do so, we shall split the sum defining $V$ into two pieces, and analyze each separately.

More precisely, define the set of *essential indices* by:

$$N(z) := \{n^-(z), n^-(z) + 1, ..., n^+(z) - 1, n^+(z)\} \setminus \{n(z) - 1, n(z), n(z) + 1\}$$

We now split $V$:

$$V(z) \simeq V_1(z) + V_2(z) = \sum_{n \in N(z)} \int_{r_{n-1}}^{r_n} \left( \log \left| 1 - \frac{z}{t} \right| - \log \left| 1 - \frac{z}{r_n} \right| \right) d\psi(t)$$

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We first estimate $V_1$ by introducing the function $L(w) := \log(1 - ze^{-\omega})$, which has the property that for each $n \in N(z)$, $L$ is analytic in a domain containing $\{\omega \mid e^\omega \in [t_{n-1}, t_n]\}$. Note that $z$ is fixed here.

Therefore, we may write a series expansion:

$$L(\omega) - L(\omega_n) = (\omega - \omega_n)L'(\omega_n) + \int_{\omega_n}^{\omega} L''(\sigma)(\omega - \sigma) d\sigma$$

$$= (\omega - \omega_n)L'(\omega_n) + Q_n(z, t);$$

here $t = e^\omega \in [t_{n-1}, t_n]$ and $e^\omega_n = r_n$.

Since $\log(r_n) = \int_{t_{n-1}}^{t_n} \log(t) d\psi(t)$, we have that:

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We directly estimate \( \sup_{t \in [t_{n-1}, t_n]} |Q_n(z, t)| \lesssim \frac{d_n^2}{|z - r_n|^2} \). Employing our assumption that \( \psi \) induces a logarithmically regular partition, we conclude \( V_1(z) \simeq 1 \).

Moving on to \( V_2 \), we write

\[
\log |z - r_n| - \log |z - z_n| = \Re \left( \int_{z_n}^{r_n} \frac{d\zeta}{\zeta - z} \right).
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Integrating along the arc \( \zeta = r_n e^{-i\theta}, \theta \in [0, \frac{cd_n}{r_n}] \), we see that:

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|\log |z - r_n| - \log |z - z_n|| \lesssim \frac{d_n^2}{|z - r_n|^2}.
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Again, by logarithmic regularity, we conclude \( V_2(z) \simeq 1 \). Thus, \( V \simeq 1 \) for \( \Im(z) \geq 0 \), so we may apply our corollary to conclude \( \Lambda(\psi) \) is sampling. This is the first claim of the theorem.
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$$|\log |z - r_n| - \log |z - z_n|| \lesssim \frac{d_n^2}{|z - r_n|^2}.$$ Again, by logarithmic regularity, we conclude $V_2(z) \simeq 1$. Thus, $V \simeq 1$ for $\Im(z) \geq 0$, so we may apply our corollary to conclude $\Lambda(\psi)$ is sampling. This is the first claim of the theorem.
We now move on to the second result of this theorem: if \( \psi'(x) = o\left(\frac{1}{x}\right) \) when \( x \to \infty \), then \( \Lambda(\psi) \) is not sampling for PW.

We will show this by exhibiting a sequence \( \{f_n\} \subset PW \) such that for \( \Lambda(\psi) = \{ \lambda_k \} \):

\[
\lim_{n \to \infty} \frac{\sum_k |f_n(\lambda_k)|^2}{\|f_n\|_{PW}^2} \to 0.
\]

Let \( \{t_n\} \) be such that \( \psi(t_n) = n \) and suppose \( n \) is sufficiently large for the following construction to work. We require \( \xi_n \in (t_n, \frac{t_n + 1}{2}) \) to be such that \( \psi(\xi_n) = n + \frac{1}{2} + \epsilon_n \), where \( \epsilon_n \) will be chosen below.
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Define a bounded, continuous $\phi_n$ with the following properties:

1. $\phi_n(t) = -t, |t| < \frac{1}{2}$.
2. $\phi_n(t) = \psi_n(t) - n - \frac{1}{2}, t_n < t < \xi_n$.
3. $\phi_n(t) = \psi(t) - n - \frac{3}{2}, 2\xi < t < t_{n+1}$.
4. Linear everywhere else.

We choose $\epsilon_n$ such that $\int_{t_n}^{\xi_n} \frac{\phi_n(x)}{x} dx = 0$. As $n \to \infty$, $\epsilon_n \to 0$.

We define a subharmonic function

$$U_n(z) := \lim_{R \to \infty} \int_{-R}^{R} \left( \log \left| 1 - \frac{z}{t} \right| \right) (1 + \phi_n'(t)) dt.$$

We directly estimate:

$$U_n(x) = \int_0^{|x|} \frac{\phi_n(t) - \frac{1}{2}}{t} dt + O(1) \text{ as } |x| \to \infty.$$
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2. $\phi_n(t) = \psi_n(t) - n - \frac{1}{2}, t_n < t < \xi_n$.
3. $\phi_n(t) = \psi(t) - n - \frac{3}{2}, 2\xi < t < t_{n+1}$.
4. Linear everywhere else.

We choose $\epsilon_n$ such that $\int_{t_n}^{t_n+1} \frac{\phi_n(x)}{x} dx = 0$. As $n \to \infty$, $\epsilon_n \to 0$.

We define a subharmonic function

$$U_n(z) := \lim_{R \to \infty} \int_{-R}^{R} \left( \log \left| 1 - \frac{z}{t} \right| \right) (1 + \phi_n'(t)) dt.$$

We directly estimate:

$$U_n(x) = \int_0^{|x|} \frac{\phi_n(t) - \frac{1}{2}}{t} dt + O(1) \text{ as } |x| \to \infty.$$
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From this and our assumption that $\phi_n'(t) = o(\frac{1}{t})$, we see there is an interval $[(1 - o(1))\xi_n, 2\xi_{2n}]$ such that $U_n(x) + \frac{1}{2} \log(x) \approx 1$ on this interval.

On the other hand, for $|x| \notin [t_n, t_{n+1}]$, $U_n(x) = -\log|x| + O(1)$.

Setting $\Omega_n := [t_n, t_{n+1}] \cup [2\xi_n, t_{n+1}]$, we see:

$$\int_{\Omega_n} e^{2U_n(x)} \, dx \to \infty \text{ but } \int_{\mathbb{R} \setminus \Omega_n} e^{2U_n(x)} \, dx < C < \infty$$

Thus, $e^{2U_n} \in L^2(\mathbb{R})$ but $\|e^{U_n}\|_2 \to \infty$. 
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Similarly, if we have a sequence of reals $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ satisfiying $\gamma_{k+1} - \gamma_k \approx 1$, we have:

$$\sum_{\gamma \in \mathbb{R} \setminus \Omega_n} e^{2U_n(\gamma_k)} < C.$$ 

Following our construction in the corollary, we define $f_n$ as follows:

$$f_n(z) := \lim_{R \to \infty} \prod_{|\gamma_k| < R} \left( 1 - \frac{z}{\gamma_k} \right)$$

where $\Gamma = \{\gamma_k\}$ is a separated real sequence such that

$$|f_n(z)| e^{-U_n(z)} \sim \min(1, \text{dist}(z, \Gamma)).$$
Similarly, if we have a sequence of reals \( \Gamma = \{ \gamma_k \}_{k=1}^\infty \) satisfying \( \gamma_{k+1} - \gamma_k \simeq 1 \), we have:

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We see $f_n$ is of exponential type $\pi$. Moreover, our earlier estimate yields $f_n \in PW$ but $\|f_n\|_{PW} \to \infty$.

Finally, notice that $\text{dist}(\lambda_k, \Gamma) \to 0$ uniformly when $n \to \infty$, and that $\gamma_k = k \ \forall k < 0$. Along with our earlier estimate, we see that:

$$\sum_k |f_n(\lambda_k)|^2 \|f_n\|_{PW}^2 \to 0.$$

This violates the sampling condition, so we have proven the second claim of main result 2.
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This violates the sampling condition, so we have proven the second claim of main result 2.


