A DISPERSIVE REGULARIZATION FOR THE MODIFIED CAMASSA-HOLM EQUATION*

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Abstract. In this paper, we present a dispersive regularization approach to construct a global N-peakon weak solution to the modified Camassa-Holm equation (mCH) in one dimension. In particular, we perform a double mollification for the system of ODEs describing trajectories of N-peakon solutions and obtain N smoothed peakons without collisions. Though the smoothed peakons do not give a solution to the mCH equation, the weak consistency allows us to take the smoothing parameter to zero and the limiting function is a global N-peakon weak solution. The trajectories of the peakons in the constructed solution are globally Lipschitz continuous and do not cross each other. When $N = 2$, the solution is a sticky peakon weak solution. At last, using the N-peakon solutions and through a mean field limit process, we obtain global weak solutions for general initial data m_0 in Radon measure space.

Key words. peakon interaction, dispersive limit, non-uniqueness, correct speed of singularity, selection principle, weak solutions

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1. Introduction. This work is devoted to investigate the N-peakon solutions to the following modified Camassa-Holm (mCH) equation with cubic nonlinearity:

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

subject to the initial condition

$$m(x, 0) = m_0(x), \quad x \in \mathbb{R}. \tag{2}$$

From the fundamental solution $G(x) = \frac{1}{2}e^{-|x|}$ to the Helmholtz operator $1 - \partial_{xx}$, function u can be written as a convolution of m with the kernel G:

$$u(x, t) = \int_{\mathbb{R}} G(x - y)m(y, t)dy.$$

In the mCH equation, the shape of function G is referred to as a peakon at $x = 0$ and the mCH equation has weak solutions (see Definition 2.2) with N peakons, which are of the form [12, 14]:

$$u^N(x, t) = \sum_{i=1}^{N} p_i G(x - x_i(t)), \quad m^N(x, t) = \sum_{i=1}^{N} p_i \delta(x - x_i(t)), \tag{3}$$

where p_i $(1 \le i \le N)$ are constant amplitudes of peakons. We call this kind of weak solutions as N-peakon solutions. When $x_1(t) < x_2(t) < \cdots < x_N(t)$, trajectories $x_i(t)$

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of N-peakon solutions in (3) satisfies [12, 14]:

$$\frac{d}{dt}x_i = \frac{1}{6}p_i^2 + \frac{1}{2}\sum_{j<i} p_i p_j e^{x_j - x_i} + \frac{1}{2}\sum_{j>i} p_i p_j e^{x_i - x_j} + \sum_{1\le m<i<n\le N} p_m p_n e^{x_m - x_n}. \tag{4}$$

In general, solutions $\{x_i(t)\}_{i=1}^N$ to (4) will collide with each other in finite time (see Remark 2.9). By the standard ODE theories, we know that (4) has global solutions $\{x_i(t)\}_{i=1}^N$ subject to any initial data $\{x_i(0)\}_{i=1}^N$. However, $u^N(x,t)$ constructed by (3) with global solutions $\{x_i(t)\}_{i=1}^N$ to (4) is not a weak solution to the mCH equation after the first collision time (see Remark 2.11). There are some nature questions:

(i) What will be a weak solution to the mCH equation after collisions? Is it unique? If not unique, what is the selection principle?

(ii) If there is a weak solution to the mCH equation after collisions, is it still in the form of N-peakon solutions?

(iii) If the weak solution is still an N-peakon solution after collision, how do peakons evolve? In other words, do they stick together, cross each other, or scatter?

Paper [12] showed global existence and nonuniqueness of weak solutions when initial data $m_0 \in \mathcal{M}(\mathbb{R})$ (Radon measure space), which partially answered question (i). In Subsection 2.2, we prove global existence of N-peakon solutions, which gives an answer to question (ii). After collision, all the situations mentioned in the above question (iii) can happen (see Remark 2.9).

In this paper, we will study these questions through a dispersive regularization for the following reasons (see (97) for the dispersive effects of our mollification method):

(i) This dispersive regularization could be a candidate for the selection principle.

(ii) As described below, if initial datum is of N-peakon form, then the regularized solution $u^{N,\epsilon}$ is also of N-peakon form, and so is the limiting N-peakon solution.

The main purpose of this paper is to study the behavior of $\epsilon \to 0$ limit for the dispersive regularization. First, we introduce the dispersive regularization for the mCH equation.

To illustrate the dispersive regularization method clearly, we start with one peakon solution $pG(x-x(t))$ (solitary wave solution). We know that $pG(x-x(t))$ is a weak solution if and only if the traveling speed is $\frac{d}{dt}x(t) = \frac{1}{6}p^2$ [12, Proposition 4.3]. Because characteristics equation for (1) is given by

$$\frac{d}{dt}x(t) = u^2(x(t),t) - u_x^2(x(t),t), \tag{5}$$

for solution $pG(x-x(t))$ we obtain

$$\frac{d}{dt}x(t) = p^2G^2(0) - p^2(G_x^2)(0) = \frac{1}{6}p^2. \tag{6}$$

Equation (6) implies that to obtain solitary wave solutions, the correct definition of G_x^2 at 0 is given by

$$(G_x^2)(0) = G^2(0) - \frac{1}{6} = \frac{1}{12}. \tag{7}$$

However, G_x^2 is a BV function which has a removable discontinuity at 0 and

$$(G_x^2)(0-) = (G_x^2)(0+) = \frac{1}{4}, \tag{8}$$

which is different with (7). To understand the discrepancy between (7) and (8), our strategy is to use the dispersive regularization and the limit of the regularization. Mollify $G(x)$ as

$$G^\epsilon(x) := (\rho_\epsilon * G)(x),$$

where ρ_ϵ is a mollifier that is even (see Definition 2.1). Then, we can obtain (7) in the limiting process (Lemma 2.5):

$$\lim_{\epsilon\to 0}(\rho_\epsilon * (G_x^\epsilon)^2)(0) = \frac{1}{12}. \tag{9}$$

The above limiting process is independent of the mollifier ρ_ϵ.

Naturally, we generalize this dispersive regularization method to N-peakon solutions $u^N(x,t) = \sum_{i=1}^N p_i G(x - x_i(t))$. From the characteristic equation (5), we formally obtain the system of ODEs for $x_i(t)$

$$\frac{d}{dt}x_i(t) = \left[u^N(x_i(t),t)\right]^2 - \left[u_x^N(x_i(t),t)\right]^2, \quad i = 1,\ldots,N. \tag{10}$$

$\left[u_x^N(x,t)\right]^2 = \left(\sum_{j=1}^N p_j G_x(x - x_j(t))\right)^2$ is a BV function and it has a discontinuity at $x_i(t)$. By using similar regularization method in (9), we regularize the vector field in (10). For $\{x_k\}_{k=1}^N$, denote

$$u^{N,\epsilon}(x;\{x_k\}) := \sum_{i=1}^N p_i G^\epsilon(x - x_i) \quad \text{and} \quad U_\epsilon^N(x;\{x_k\}) := \left[u^{N,\epsilon}\right]^2 - \left[u_x^{N,\epsilon}\right]^2. \tag{11}$$

The dispersive regularization for N peakons is given by

$$\frac{d}{dt}x_i^\epsilon(t) = U^{N,\epsilon}(x_i^\epsilon(t);\{x_k^\epsilon(t)\}) := (\rho_\epsilon * U_\epsilon^N)(x_i^\epsilon(t);\{x_k^\epsilon(t)\}), \qquad i = 1,\ldots,N. \tag{12}$$

The above regularization method is subtle. We emphasize that if we use U_ϵ^N given by (11) as a vector field (which is already globally Lipschitz continuous) instead of $U^{N,\epsilon}$, then comparing with (9) we have

$$\lim_{\epsilon\to 0}(G_x^\epsilon)^2(0) = 0.$$

In this case, the traveling speed of the soliton (one peakon) is given by

$$\frac{d}{dt}x(t) = p^2G^2(0) - p^2(G_x^2)(0) = \frac{1}{4}p^2,$$

which is different with the correct speed $\frac{1}{6}p^2$ for one peakon solution.

By solutions to (12), we construct approximate N-peakon solutions to (1) as:

$$u^{N,\epsilon}(x,t) := \sum_{i=1}^N p_i G^\epsilon(x - x_i^\epsilon(t)).$$

Let $\epsilon \to 0$ in $u^{N,\epsilon}(x,t)$ and we can obtain an N-peakon solution

$$u^N(x,t) = \sum_{i=1}^N p_i G(x - x_i(t)), \tag{13}$$

to the mCH equation, where $x_i(t)$ are Lipschitz functions (see Theorem 2.4).

If we fix N and let ϵ go to 0 in the regularized system of ODEs (12), we can obtain a limiting ($\epsilon \to 0$ in the sense described in Proposition 2.7) system of ODEs to describe N-peakon solutions, $i = 1, 2, \cdots, N$,

(14)
$$\frac{d}{dt}x_i(t) = \left(\sum_{j=1}^{N} p_j G(x_i(t) - x_j(t))\right)^2 - \left(\sum_{j\in\mathcal{N}_{i1}(t)} p_j G_x(x_i(t) - x_j(t))\right)^2 - \frac{1}{12}\left(\sum_{k\in\mathcal{N}_{i2}(t)} p_k\right)^2.$$

Here $\mathcal{N}_{i1}(t)$ and $\mathcal{N}_{i2}(t)$, $i = 1, 2, \cdots, N$, are defined by (42). The vector field of the above system is not Lipschitz continuous. Solutions for this equation are not unique, which implies peakon solutions to (1) are not unique (see Remark 2.9). Indeed, the nonuniqueness of peakon solutions was also obtained in [12]. When $x_1(t) < x_2(t) < \cdots < x_N(t)$, the system of ODEs (14) is equivalent to (4).

We also prove that trajectories $x_i^\epsilon(t)$ given by (12) never collide with each other (see Theorem 3.2), which means if $x_1^\epsilon(0) < x_2^\epsilon(0) < \cdots < x_N^\epsilon(0)$, then $x_1^\epsilon(t) < x_2^\epsilon(t) < \cdots < x_N^\epsilon(t)$ for any $t > 0$. For the limiting N-peakon solutions (13), we have $x_1(t) \le x_2(t) \le \cdots \le x_N(t)$. Notice that the sticky N-peakon solutions obtained in [12] also have this property and in the sticky N-peakon solutions, $\{x_i(t)\}_{i=1}^N$ stick together whenever they collide. When $N = 2$, we prove that peakon solutions given by the dispersive regularization are exactly the sticky peakon solutions (see Theorem 3.4). However, the situation when $N \ge 3$ can be more complicated. Some of the peakon solutions given by the dispersive regularization are sticky peakon solutions (see Figure 1) and some are not (see Figure 2).

For general initial data $m_0 \in \mathcal{M}(\mathbb{R})$, we use a mean field limit method to prove global existence of weak solutions to (1) (see Section 4).

There are also some other interesting properties about the mCH equation, which we list below.

The mCH equation was introduced as a new integrable system by several different researchers [8, 10, 22, 23]. The mCH equation has a bi-Hamiltonian structure [14, 22] with Hamiltonian functionals

$$H_0(m) = \int_{\mathbb{R}} mu\,dx, \quad H_1(m) = \frac{1}{4}\int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) dx. \tag{15}$$

Equation (1) can be written in the bi-Hamiltonian form [14, 22],

$$m_t = -((u^2 - u_x^2)m)_x = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m},$$

where

$$J = -\partial_x\left(m\partial_x^{-1}(m\partial_x)\right), \quad K = \partial_x^3 - \partial_x$$

are compatible Hamiltonian operators. Here H_0 and H_1 are conserved quantities for smooth solutions. H_0 is also a conserved quantity for $W^{2,1}(\mathbb{R})$ weak solutions [12]. N-peakon solutions are not in the solution class $W^{2,1}(\mathbb{R})$ and H_0, H_1 are not conserved for N-peakon solutions in the case $N \ge 2$; see Remark 2.9 for the case $N = 2$. This is different with the Camassa-Holm equation [3]:

$$m_t + (um)_x + mu_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$

which also has N-peakon solutions of the form

$$u^N(x,t) = \sum_{i=1}^{N} p_i(t)e^{-|x-x_i(t)|}.$$

The amplitude $p_i(t)$ evolves with time which is different with the N-peakon solutions to mCH equation (1) where p_i are constants. $p_i(t)$ and $x_i(t)$ satisfy the following Hamiltonian system of ODEs:

$$(16)\qquad \begin{cases} \dfrac{d}{dt}x_i(t) = \displaystyle\sum_{j=1}^{N} p_j(t)e^{-|x_i(t)-x_j(t)|}, \quad i=1,\dots,N, \\ \dfrac{d}{dt}p_i(t) = \displaystyle\sum_{j=1}^{N} p_i(t)p_j(t)\mathrm{sgn}\big(x_i(t)-x_j(t)\big)e^{-|x_i(t)-x_j(t)|}, \quad i=1,\dots,N, \end{cases}$$

and the Hamiltonian function is given by

$$\mathcal{H}_0(t) = \frac{1}{2}\sum_{i,j=1}^{N} p_i(t)p_j(t)e^{-|x_i(t)-x_j(t)|},$$

which is a conserved quantity for N-peakon solutions and the corresponding functional H_0 given by (15) is conserved for smooth solutions for the Camassa-Holm equation. When $p_i(0) > 0$, there is no collision between $x_i(t)$ [4, 6, 18]. Hence, solutions to system (16) exist globally. However, collisions may occur if $p_i(0)$'s have opposite signs. In [16], Holden and Raynaud studied this case and they constructed a new set of ordinary differential equations which is well-posedness even when collisions occur. They obtained global N-peakon solutions to the Camassa-Holm equation, which conserve the Hamiltonian $\mathcal{H}_0$. For more details about peakon solutions to the Camassa-Holm equation, one can also refer to[1, 2, 7, 13, 17].

In comparison, system (4) is a nonautonomous Hamiltonian system as described below. Let $\tilde{x}_i(t) := x_i(t) - \frac{1}{6}p_i^2 t$. Denote

$$X(t) := (\tilde{x}_1(t), \tilde{x}_2(t), \cdots, \tilde{x}_N(t))^T,$$

and

$$\mathcal{H}(X,t) := \sum_{1\le i<j\le N} p_i p_j e^{x_i(t)-x_j(t)} = \sum_{1\le i<j\le N} p_i p_j e^{\frac{1}{6}(p_j^2-p_i^2)t+\tilde{x}_i(t)-\tilde{x}_j(t)}.$$

Then, (4) can be rewritten as a Hamiltonian system:

$$(17)\qquad \frac{dX}{dt} = A\frac{\delta\mathcal{H}}{\delta X},$$

where

$$(18)\qquad A=(a_{ij})_{N\times N}, \quad a_{ij} = \begin{cases} -\frac{1}{2}, & i<j; \\ 0, & i=j; \\ \frac{1}{2}, & i>j. \end{cases}, \text{ and } \frac{\delta\mathcal{H}}{\delta X} := \Big(\frac{\partial\mathcal{H}}{\partial\tilde{x}_1},\dots,\frac{\partial\mathcal{H}}{\partial\tilde{x}_N}\Big).$$

Notice that $\mathcal{H}$ depends on t and it is not a conservative quantity.

For more results about local well-posedness and blow up behavior of the strong solutions to (1) one can refer to [5, 9, 14, 15, 21]. In [24], Zhang used the method of dissipative approximation to prove the existence and uniqueness of global entropy weak solutions u in $W^{2,1}(\mathbb{R})$ for the mCH equation (1).

The rest of this article is organized as follows. In Section 2, we introduce the dispersive regularization in detail and prove global existence of N-peakon solutions.

By a limiting process, we obtain a system of ODEs to describe N-peakon solutions. In Section 3, we prove that trajectories of N-peakon solutions given by dispersive regularization will never cross each other. When $N = 2$, the limiting peakon solutions are exactly the sticky peakon solutions. When $N = 3$, we present two figures to show two different situations. In Section 4, we use a mean field limit method to prove global existence of weak solutions to (1) for general initial data $m_0 \in \mathcal{M}(\mathbb{R})$. At last, we use the same double mollification method to mollify the mCH equation directly. By linearizing the modified equation, we show that this regularization has the dispersive effects.

2. Dispersive regularization and N-peakon solutions.

In this section, we introduce the dispersive regularization in details and use the regularized ODE system to give approximate solutions. Then, by some compactness arguments we prove global existence of N-peakon solutions.

2.1. Dispersive regularization and weak consistency.

First, we use smooth functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ to define mollifiers. $f \in \mathcal{S}(\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R})$ and for all positive integers m and n

$$\sup_{x\in\mathbb{R}} |x^m f^{(n)}(x)| < \infty.$$

DEFINITION 2.1. (i). *Define the mollifier* $0 \le \rho \in \mathcal{S}(\mathbb{R})$ *satisfying*

$$\int_{\mathbb{R}} \rho(x)dx = 1, \quad \rho(x) = \rho(|x|) \ \ \textit{for} \ \ x \in \mathbb{R}.$$

(ii). *For each* $\epsilon > 0$, *set*

$$\rho_\epsilon(x) := \frac{1}{\epsilon}\rho(\frac{x}{\epsilon}).$$

Fix an integer $N > 0$. Give an initial data

$$m_0^N(x) = \sum_{i=1}^N p_i\delta(x - c_i), \quad c_1 < c_2 < \cdots < c_N \ \ \text{and} \ \ \sum_{i=1}^N |p_i| \le M_0, \tag{19}$$

for some constants p_i, c_i $(1 \le i \le N)$ and M_0.

As stated in Introduction, we set $G^\epsilon(x) = (G * \rho_\epsilon)(x)$. For any N particles $\{x_k\}_{k=1}^N \subset \mathbb{R}$, define ($p_k$ is the same as in (19))

$$u^{N,\epsilon}(x; \{x_k\}_{k=1}^N) := \sum_{k=1}^N p_k G^\epsilon(x - x_k),$$

$$U_\epsilon^N(x; \{x_k\}_{k=1}^N) := \left[(u^{N,\epsilon})^2 - (\partial_x u^{N,\epsilon})^2\right](x; \{x_k\}_{k=1}^N),$$

and

$$U^{N,\epsilon}(x; \{x_k\}_{k=1}^N) := (\rho_\epsilon * U_\epsilon^N)(x; \{x_k\}_{k=1}^N).$$

The system of ODEs for dispersive regularization is given by

$$\frac{d}{dt}x_i^\epsilon(t) = U^{N,\epsilon}(x_i^\epsilon(t); \{x_k^\epsilon(t)\}_{k=1}^N), \quad i = 1, \cdots, N, \tag{20}$$

with initial data $x_i^\epsilon(0) = c_i$ given in (19). This system is equivalent to (12) mentioned in Introduction. Because $U^{N,\epsilon}$ is Lipschitz continuous and bounded, existence and uniqueness of a global solution $\{x_i^\epsilon(t)\}_{i=1}^N$ to this system of ODEs follow from standard ODE theories. By using the solution $\{x_i^\epsilon(t)\}_{i=1}^N$, we set

$$u^{N,\epsilon}(x,t) := u^{N,\epsilon}(x; \{x_k^\epsilon(t)\}_{k=1}^N), \tag{21}$$

and

$$m^{N,\epsilon}(x,t) := \sum_{i=1}^N p_i \rho_\epsilon(x - x_i^\epsilon(t)), \quad m_\epsilon^N(x,t) := \sum_{i=1}^N p_i \delta(x - x_i^\epsilon(t)). \tag{22}$$

Due to $(1-\partial_{xx})G^\epsilon = \rho_\epsilon$, we have

$$m^{N,\epsilon}(x,t) = (\rho_\epsilon * m_\epsilon^N)(x,t) \ \text{ and } \ (1-\partial_{xx})u^{N,\epsilon}(x,t) = m^{N,\epsilon}(x,t). \tag{23}$$

Set

$$U_\epsilon^N(x,t) := U_\epsilon^N(x; \{x_k^\epsilon(t)\}_{k=1}^N), \quad U^{N,\epsilon}(x,t) := U^{N,\epsilon}(x; \{x_k^\epsilon(t)\}_{k=1}^N). \tag{24}$$

Therefore, $U^{N,\epsilon}(x,t) = (\rho_\epsilon * U_\epsilon^N)(x,t)$ and (20) (or (12)) can be rewritten as

$$\frac{d}{dt} x_i^\epsilon(t) = U^{N,\epsilon}(x_i^\epsilon(t), t), \quad i = 1, \cdots, N. \tag{25}$$

Next, we show that $u^{N,\epsilon}$ defined by (21) is weak consistent with the mCH equation (1). Let us give the definition of weak solutions first. Rewrite (1) as an equation of u,

$$\begin{aligned} &(1-\partial_{xx})u_t + [(u^2 - u_x^2)(u - u_{xx})]_x \\ &= (1-\partial_{xx})u_t + (u^3 + uu_x^2)_x - \frac{1}{3}(u^3)_{xxx} + \frac{1}{3}(u_x^3)_{xx} = 0. \end{aligned}$$

For a test function $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$ $(T > 0)$, we denote the functional

$$\begin{aligned} \mathcal{L}(u,\phi) :=& \int_0^T \int_{\mathbb{R}} u(x,t)[\phi_t(x,t) - \phi_{txx}(x,t)]dxdt \\ &- \frac{1}{3}\int_0^T \int_{\mathbb{R}} u_x^3(x,t)\phi_{xx}(x,t)dxdt - \frac{1}{3}\int_0^T \int_{\mathbb{R}} u^3(x,t)\phi_{xxx}(x,t)dxdt \\ &+ \int_0^T \int_{\mathbb{R}} (u^3 + uu_x^2)\phi_x(x,t)dxdt. \end{aligned} \tag{26}$$

Then, the definition of weak solutions in terms of u is given as follows.

Definition 2.2. *For $m_0 \in \mathcal{M}(\mathbb{R})$, a function*

$$u \in C([0,T); H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$$

is said to be a weak solution of the mCH equation if

$$\mathcal{L}(u,\phi) = -\int_{\mathbb{R}} \phi(x,0)dm_0$$

holds for all $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$. If $T = +\infty$, we call u as a global weak solution of the mCH equation.

For simplicity, we denote

$$\langle f(x,t), g(x,t)\rangle := \int_0^\infty \int_{\mathbb{R}} f(x,t)g(x,t)dxdt.$$

With the definitions (22)-(25), for any $\phi \in C_c^\infty(\mathbb{R}\times[0,T))$, we have

$$\begin{aligned}
\langle m_\epsilon^N, \phi_t\rangle + \langle U^{N,\epsilon} m_\epsilon^N, \phi_x\rangle &= \int_0^T \int_{\mathbb{R}} \sum_{i=1}^N p_i \delta(x - x_i^\epsilon(t))\phi_t(x,t)dxdt \\
&+ \int_0^T \int_{\mathbb{R}} \sum_{i=1}^N p_i \delta(x - x_i^\epsilon(t)) U^{N,\epsilon}(x,t)\phi_x(x,t)dxdt \\
&= \int_0^T \sum_{i=1}^N p_i [\phi_t(x_i^\epsilon(t),t) + U^{N,\epsilon}(x_i^\epsilon(t),t)\phi_x(x_j^\epsilon(t),t)]dt \\
&= \int_0^T \sum_{i=1}^N p_i \frac{d}{dt}\phi(x_i^\epsilon(t),t)dt = -\sum_{i=1}^N \phi(x_i(0),0)p_i = -\int_{\mathbb{R}} \phi(x,0)dm_0^N. \qquad (27)
\end{aligned}$$

On the other hand, combining the definition (23) and (26) gives

$$\begin{aligned}
\mathcal{L}(u^{N,\epsilon},\phi) &= \int_0^T \int_{\mathbb{R}} u^{N,\epsilon}[\phi_t - \phi_{txx}]dxdt - \frac{1}{3}\int_0^T \int_{\mathbb{R}} (\partial_x u^{N,\epsilon})^3 \phi_{xx} dxdt \\
&\quad - \frac{1}{3}\int_0^T \int_{\mathbb{R}} (u^{N,\epsilon})^3 \phi_{xxx} dxdt + \int_0^T \int_{\mathbb{R}} ((u^{N,\epsilon})^3 + u^\epsilon (u_x^{N,\epsilon})^2)\phi_x dxdt \\
&= \langle \phi_t, (1-\partial_{xx})u^{N,\epsilon}\rangle + \langle [(u^{N,\epsilon})^2 - (\partial_x u^{N,\epsilon})^2](1-\partial_{xx})u^{N,\epsilon}, \phi_x\rangle \\
&= \langle m^{N,\epsilon}, \phi_t\rangle + \langle U_\epsilon^N m^{N,\epsilon}, \phi_x\rangle.
\end{aligned}$$

Set

$$\begin{aligned}
E_{N,\epsilon} :&= \mathcal{L}(u^{N,\epsilon},\phi) + \int_{\mathbb{R}} \phi(x,0)dm_0^N \\
&= \langle m^{N,\epsilon} - m_\epsilon^N, \phi_t\rangle + \langle U_\epsilon^N m^{N,\epsilon} - U^{N,\epsilon} m_\epsilon^N, \phi_x\rangle. \qquad (28)
\end{aligned}$$

We have the following consistency result.

PROPOSITION 2.3. *We have the following estimate for $E_{N,\epsilon}$ defined by (28):*

$$|E_{N,\epsilon}| \le C\epsilon, \qquad (29)$$

where the constant C is independent of N, ϵ.

Proof. By changing of variable and the definition of Schwartz function, we can obtain

$$\int_{\mathbb{R}} |x|\rho_\epsilon(x)dx = \int_{\mathbb{R}} |x|\frac{1}{\epsilon}\rho(\frac{x}{\epsilon})dx = \epsilon \int_{\mathbb{R}} |x|\rho(x)dx \le C_\rho \cdot \epsilon, \qquad (30)$$

for some constant C_ρ.

Due to $\sum_{i=1}^N |p_i| \le M_0$ and (30), the first term on the right hand side of (28) can be estimated as

$$\begin{aligned}\left|\langle m^{N,\epsilon} - m_\epsilon^N, \phi_t\rangle\right| &= \left|\int_0^T \int_{\mathbb{R}} \sum_{i=1}^N p_i \rho_\epsilon(x - x_i^\epsilon(t))[\phi_t(x,t) - \phi_t(x_i^\epsilon(t),t)]dxdt\right| \\ &\le \sum_{i=1}^N |p_i| \int_0^T \int_{\mathbb{R}} \rho_\epsilon(x - x_i^\epsilon(t))||\phi_{tx}||_{L^\infty}|x - x_i^\epsilon(t)|dxdt \\ &\le C_\rho M_0 ||\phi_{tx}||_{L^\infty} T\epsilon.\end{aligned}$$

For the second term, by definitions (22) and (24) we can obtain

$$\begin{aligned}&\langle U_\epsilon^N m^{N,\epsilon} - U^{N,\epsilon} m_\epsilon^N, \phi_x\rangle \\ &= \sum_{i=1}^N p_i \int_0^T \int_{\mathbb{R}} U_\epsilon^N(x)\rho_\epsilon(x - x_i^\epsilon(t))\phi_x(x,t)dxdt - \sum_{i=1}^N p_i \int_0^T U^{N,\epsilon}(x_i^\epsilon(t))\phi_x(x_i^\epsilon(t),t)dt \\ &= \sum_{i=1}^N p_i \int_0^T \int_{\mathbb{R}} U_\epsilon^N(x)\rho_\epsilon(x - x_i^\epsilon(t))\phi_x(x,t)dxdt \\ &\qquad - \sum_{i=1}^N p_i \int_0^T \int_{\mathbb{R}} U_\epsilon^N(x)\rho_\epsilon(x_i^\epsilon(t) - x)\phi_x(x_i^\epsilon(t),t)dxdt \\ &= \sum_{i=1}^N p_i \int_0^T \int_{\mathbb{R}} U_\epsilon^N(x)\rho_\epsilon(x - x_i^\epsilon(t))[\phi_x(x,t) - \phi_x(x_i^\epsilon(t),t)]dxdt.\end{aligned}$$

∎

Due to $||U_\epsilon^N||_{L^\infty} \le \frac{1}{2}M_0^2$, we have

$$\left|\langle U_\epsilon^N m^{N,\epsilon} - U^{N,\epsilon} m_\epsilon^N, \phi_x\rangle\right| \le \frac{1}{2} C_\rho M_0^3 ||\phi_{xx}||_{L^\infty} T\epsilon.$$

This ends the proof.

Notice that

$$(1 - \partial_{xx})G^\epsilon = \rho_\epsilon.$$

The mollification approximates the Dirac delta function with a 'blob function' ρ_ϵ, which shares some ideas with the traditional blob regularization for vortex sheet [19]. However, our regularization is more than 'blob regularization' and the key feature is the double mollification that guarantees the weak consistency. If we use

$$\frac{d}{dt} x_i^\epsilon(t) = U_\epsilon^N(x_i^\epsilon(t); \{x_k\}_{k=1}^N)$$

to define approximate trajectories instead of (20), we will not get the weak consistency result. Regarding this issue, one can refer to the discussion in Introduction or Lemma 2.5. In Section 5, we find that this regularization has the dispersive effects by studying the modified equation, which justifies 'dispersive regularization' in the title.

2.2. Convergence theorem. In this subsection, we prove global existence of N-peakon solutions for the mCH equation and this answers the second question (ii) in Introduction.

THEOREM 2.4. *Let $m_0^N(x)$ be given by (19) and $\{x_i^\epsilon(t)\}_{i=1}^N$ is defined by (25) subject to initial data $x_i^\epsilon(0) = c_i$. $u^{N,\epsilon}(x,t)$ is defined by (21). Then, the following holds.*

(i). *There exist* $\{x_i(t)\}_{i=1}^N \subset C([0,+\infty))$*, such that* $x_i^\epsilon \to x_i$ *in* $C([0,T])$ *as* $\epsilon \to 0$ *(in subsequence sense) for any* $T>0$*. Moreover,* $x_i(t)$ *is globally Lipschitz continuous and for a.e.* $t>0$*, we have*

$$\left|\frac{d}{dt}x_i(t)\right| \leq \frac{1}{2}M_0^2 \quad \text{for} \quad i=1,\dots,N. \tag{31}$$

(ii). *Set*

$$u^N(x,t) := \sum_{i=1}^N p_i G(x-x_i(t)), \tag{32}$$

and we have (in subsequence sense)

$$u^{N,\epsilon} \to u^N, \quad \partial_x u^{N,\epsilon} \to u_x^N \text{ in } L^1_{loc}(\mathbb{R}\times[0,+\infty)) \text{ as } \epsilon \to 0. \tag{33}$$

(iii). $u^N(x,t)$ *is an* N*-peakon solution to* (1)*.*

Proof. (i). Due to $G^\epsilon = G * \rho_\epsilon$, we have

$$||G^\epsilon||_{L^\infty} \leq \frac{1}{2} \text{ and } ||G_x^\epsilon||_{L^\infty} \leq \frac{1}{2}.$$

Hence,

$$||u^{N,\epsilon}||_{L^\infty} \leq \frac{1}{2}M_0 \text{ and } ||u_x^{N,\epsilon}||_{L^\infty} \leq \frac{1}{2}M_0, \tag{34}$$

where M_0 is given in (19). By Definition (24) and (34), we have

$$\begin{aligned} |U^{N,\epsilon}(x,t)| &\leq ||U_\epsilon^N||_{L^\infty}\int_{\mathbb{R}}\rho_\epsilon(x)dx \leq ||u^{N,\epsilon}||_{L^\infty}^2 + ||\partial_x u^{N,\epsilon}||_{L^\infty}^2 \\ &\leq \frac{1}{4}M_0^2 + \frac{1}{4}M_0^2 = \frac{1}{2}M_0^2. \end{aligned} \tag{35}$$

Combining (25) and (35), we have

$$\begin{aligned} |x_i^\epsilon(t) - x_i^\epsilon(s)| &= \left|\int_s^t \frac{d}{d\tau}x_i^\epsilon(\tau)d\tau\right| = \left|\int_s^t U^{N,\epsilon}(x_i^\epsilon(\tau),\tau)d\tau\right| \\ &\leq \int_s^t |U^{N,\epsilon}(x_i^\epsilon(\tau),\tau)|d\tau \leq \frac{1}{2}M_0^2|t-s|. \end{aligned} \tag{36}$$

For each $1\leq i\leq N$, by (35) and (36), we know $\{x_i^\epsilon(t)\}_{\epsilon>0}$ is uniformly (in ϵ) bounded and equi-continuous in $[0,T]$. For any fixed time $T>0$, Arzelà-Ascoli theorem implies that there exists a function $x_i \in C([0,T])$ and a subsequence $\{x_i^{\epsilon_k}\}_{k=1}^\infty \subset \{x_i^\epsilon\}_{\epsilon>0}$, such that $x_i^{\epsilon_k} \to x_i$ in $C([0,T])$ as $k\to\infty$. Then, use a diagonalization argument with respect to $T=1,2,\dots$ and we obtain a subsequence (still denoted as x_i^ϵ) of x_i^ϵ such that $x_i^\epsilon \to x_i$ in $C([0,T])$ as $\epsilon\to 0$ for any $T>0$. Moreover, by (36), we have

$$|x_i(t)-x_i(s)| \leq \frac{1}{2}M_0^2|t-s|.$$

Hence, $x_i(t)$ is a globally Lipschitz function and (31) holds.

(ii). Because $u^{N,\epsilon}(x,t) \to u^N(x,t)$ and $\partial_x u^{N,\epsilon}(x,t) \to u_x^N(x,t)$ as $\epsilon \to 0$ for a.e. $(x,t) \in \mathbb{R} \times [0,+\infty)$ (for $(x,t) \neq (x_i(t),t)$), then (33) follows by Lebesgue dominated convergence theorem.

(iii). Next, we prove that u^N given by (32) is a weak solution to the mCH equation.

Obviously, we have

$$u^N \in C([0,+\infty); H^1(\mathbb{R})) \cap L^\infty(0,+\infty; W^{1,\infty}(\mathbb{R})).$$

Similarly as (27), for any test function $\phi \in C_c^\infty(\mathbb{R} \times [0,\infty))$ we have

$$\langle m_\epsilon^N, \phi_t \rangle + \langle U^{N,\epsilon} m_\epsilon^N, \phi_x \rangle = -\int_{\mathbb{R}} \phi(x,0) dm_0^N,$$

where $(m_\epsilon^N,\ m^{N,\epsilon})$ is defined by (22) and $(U_\epsilon^N,\ U^{N,\epsilon})$ is defined by (24). By the consistency result (29), we have

$$\mathcal{L}(u^{N,\epsilon}, \phi) + \int_{\mathbb{R}} \phi(x,0) dm_0^N \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{37}$$

where

$$\begin{aligned} \mathcal{L}(u^{N,\epsilon}, \phi) = &\int_0^T \int_{\mathbb{R}} u^{N,\epsilon}(\phi_t - \phi_{txx}) dxdt - \frac{1}{3}\int_0^T \int_{\mathbb{R}} (\partial_x u^{N,\epsilon})^3 \phi_{xx} dxdt \\ &- \frac{1}{3}\int_0^T \int_{\mathbb{R}} (u^{N,\epsilon})^3 \phi_{xxx} dxdt + \int_0^T \int_{\mathbb{R}} [(u^{N,\epsilon})^3 + u^{N,\epsilon}(\partial_x u^{N,\epsilon})^2] \phi_x dxdt. \end{aligned} \tag{38}$$

(Here, T satisfies $\operatorname{supp}\{\phi\} \subset \mathbb{R} \times [0,T)$.) We now consider convergence for each term of $\mathcal{L}(u^{N,\epsilon}, \phi)$.

For the first term on the right hand side of (38), using (33) and the fact that $\operatorname{supp}\{\phi\}$ is compact we can see

$$\int_0^T \int_{\mathbb{R}} u^{N,\epsilon}(\phi_t - \phi_{txx}) dxdt \to \int_0^T \int_{\mathbb{R}} u^N(\phi_t - \phi_{txx}) dxdt \quad \text{as} \quad \epsilon \to 0.$$

The second term can be estimated as follows

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}} [(\partial_x u^{N,\epsilon})^3 - (u_x^N)^3] \phi_{xx} dxdt \right| \\ = &\left| \int_0^T \int_{\mathbb{R}} (\partial_x u^{N,\epsilon} - u_x^N)[(\partial_x u^{N,\epsilon})^2 + (u_x^N)^2 + \partial_x u^{N,\epsilon} u_x^N] \phi_{xx} dxdt \right| \\ \leq &\frac{3}{4} M_0^2 ||\phi_{xx}||_{L^\infty} \int\int_{\operatorname{supp}\{\phi\}} |\partial_x u^{N,\epsilon} - u_x^N| dxdt \to 0 \quad \text{as} \quad \epsilon \to 0. \end{aligned}$$

Similarly, we have the following estimates for the rest terms on the right hand side of (38):

$$\int_0^T \int_{\mathbb{R}} [(u^{N,\epsilon})^3 - (u^N)^3] \phi_{xxx} dxdt \to 0 \quad \text{as} \quad \epsilon \to 0,$$

$$\int_0^T \int_{\mathbb{R}} [(u^{N,\epsilon})^3 - (u^N)^3] \phi_x dxdt \to 0 \quad \text{as} \quad \epsilon \to 0,$$

and

$$\int_0^T\int_{\mathbb{R}}[u^{N,\epsilon}(\partial_x u^{N,\epsilon})^2 - u^N(u_x^N)^2]\phi_x dxdt$$
$$= \int_0^T\int_{\mathbb{R}}[(u^{N,\epsilon}-u^N)(\partial_x u^{N,\epsilon})^2 + u^N(\partial_x u^{N,\epsilon}+u_x^N)(\partial_x u^{N,\epsilon}-u_x^N)]\phi_x dxdt$$
$$\to 0 \quad \text{as} \quad \epsilon \to 0.$$

Hence, the above estimates shows that for any test function $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$

$$\mathcal{L}(u^{N,\epsilon},\phi) \to \mathcal{L}(u^N,\phi) \quad \text{as} \quad \epsilon \to 0. \tag{39}$$

Therefore, combining (37) and (39) gives

$$\mathcal{L}(u^N,\phi) + \int_{\mathbb{R}}\phi(x,0)dm_0^N = 0,$$

which implies that $u^N(x,t)$ is an N-peakon solution to the mCH equation with initial date $m_0^N(x)$. ∎

2.3. A limiting system of ODEs as $\epsilon \to 0$. In this section, we derive a system of ODEs to describe N-peakon solutions by letting $\epsilon \to 0$ in (25). First, we give an important lemma.

LEMMA 2.5. *The following equality holds*

$$\lim_{\epsilon\to 0}(\rho_\epsilon * (G_x^\epsilon)^2)(0) = \frac{1}{12}.$$

Proof. Set $F(y) = \int_{-\infty}^y \rho(x)dx$. Because ρ is an even function, we have

$$F(-y) = \int_{-\infty}^{-y}\rho(x)dx = \int_y^\infty \rho(x)dx.$$

Therefore,

$$F(y)+F(-y) = \int_{-\infty}^{y}\rho(x)dx + \int_y^\infty \rho(x)dx = 1. \tag{40}$$

Furthermore, we have

$$F(+\infty)=1, \quad F(-\infty)=0.$$

Due to $\rho_\epsilon(x)=\rho_\epsilon(-x)$, we can obtain

$$I_\epsilon := (\rho_\epsilon * (G_x^\epsilon)^2)(0) = \int_{\mathbb{R}}\rho_\epsilon(y)\left(\int_{\mathbb{R}}\frac{1}{2}e^{-|x-y|}\rho_\epsilon'(x)dx\right)^2 dy$$
$$= \frac{1}{4}\int_{\mathbb{R}}\rho(y)\left(\frac{1}{\epsilon}\int_{-\infty}^y e^{\epsilon(x-y)}\rho'(x)dx + \frac{1}{\epsilon}\int_y^\infty e^{\epsilon(y-x)}\rho'(x)dx\right)^2 dy$$
$$= \frac{1}{4}\int_{\mathbb{R}}\rho(y)\left(\int_{-\infty}^y e^{-\epsilon|x-y|}\rho(x)dx - \int_y^\infty e^{-\epsilon|x-y|}\rho(x)dx\right)^2 dy.$$

Then, by using Lebesgue dominated convergence theorem and (40) we have

$$\begin{aligned}
\lim_{\epsilon\to 0} I_\epsilon &= \frac{1}{4}\int_{\mathbb{R}} \rho(y)\left(\int_{-\infty}^{y}\rho(x)dx - \int_{y}^{\infty}\rho(x)dx\right)^2 dy \\
&= \frac{1}{4}\int_{\mathbb{R}} \rho(y)(F(y)-F(-y))^2 dy = \frac{1}{4}\int_{-\infty}^{\infty} F'(y)(1-2F(y))^2 dy \\
&= \frac{1}{4}\int_{-\infty}^{\infty} F'(y) - 2(F^2(y))' + \frac{4}{3}(F^3(y))' dy \\
&= \frac{1}{4}\left(F(+\infty) - 2F^2(+\infty) + \frac{4}{3}F^3(+\infty)\right) = \frac{1}{12}.
\end{aligned}$$

∎

Remark 2.6. The above limit is independent of the mollifier ρ and intrinsic to the mCH equation (1). Consider one peakon solution $pG(x-x(t))$. To obtain the correct speed for $x(t)$, the right value for G_x^2 at 0 is the limit obtained by Lemma 2.5:

$$(G_x^2)(0) = \frac{1}{12}.$$

By the jump condition for piecewise smooth weak solutions to (1) in [11, Equation (2.2)], the speed for $x(t)$ should be

$$\frac{dx(t)}{dt} = G^2(0) - \frac{1}{3}[G_x^2(0+) + G_x(0+)G_x(0-) + G_x^2(0-)],$$

implying that the correct value of G_x^2 at 0 is

$$\frac{1}{3}[G_x^2(0+) + G_x(0+)G_x(0-) + G_x^2(0-)] = \frac{1}{12},$$

which agrees with the limit obtained by Lemma 2.5. This is different from the precise representative of the BV function G_x^2 at the discontinuous point 0

$$\frac{1}{2}[G_x^2(0-) + G_x^2(0+)] = \frac{1}{4}.$$

Next, we use Lemma 2.5 to obtain the system of ODEs to describe N-peakon solutions by letting $\epsilon \to 0$ in (25).

PROPOSITION 2.7. *For any constants $\{p_i\}_{i=1}^N$, $\{x_i\}_{i=1}^N \subset \mathbb{R}$ (note that x_i are fixed compared with $x_i^\epsilon(t)$ in (21)), denote $\mathcal{N}_{i1} := \{1 \le j \le N : x_j \ne x_i\}$ and $\mathcal{N}_{i2} := \{1 \le j \le N : x_j = x_i\}$ for $1 \le i \le N$. Set*

$$u^{N,\epsilon}(x) := \sum_{j=1}^{N} p_j G^\epsilon(x - x_j),$$

and

$$U^\epsilon(x) := [\rho_\epsilon * (u^{N,\epsilon})^2](x) - [\rho_\epsilon * (u_x^{N,\epsilon})^2](x).$$

(Note that x_i are constants in $U^\epsilon(x)$ comparing with $U^{N,\epsilon}(x,t)$ defined by (24).) Then we have

(41)

$$\lim_{\epsilon\to 0} U^\epsilon(x_i) = \left(\sum_{j=1}^{N} p_j G(x_i - x_j)\right)^2 - \left(\sum_{j\in\mathcal{N}_{i1}} p_j G_x(x_i - x_j)\right)^2 - \frac{1}{12}\left(\sum_{k\in\mathcal{N}_{i2}} p_k\right)^2.$$

Proof. See appendix. ∎

Remark 2.8 (System of ODEs). From Proposition 2.7, we give a system of ODEs to describe N-peakon solution $u^N(x,t)=\sum_{i=1}^N p_iG(x-x_i(t))$. For $1\le i\le N$, set

$$\mathcal{N}_{i1}(t):=\{1\le j\le N: x_j(t)\neq x_i(t)\} \text{ and } \mathcal{N}_{i2}(t):=\{1\le j\le N: x_j(t)=x_i(t)\}. \tag{42}$$

The system of ODEs is given by, $1\le i\le N$,

$$\frac{d}{dt}x_i(t)=\left(\sum_{j=1}^N p_jG(x_i(t)-x_j(t))\right)^2-\left(\sum_{j\in\mathcal{N}_{i1}(t)} p_jG_x(x_i(t)-x_j(t))\right)^2-\frac{1}{12}\left(\sum_{k\in\mathcal{N}_{i2}(t)} p_k\right)^2. \tag{43}$$

Before the collisions of peakons, we can deduce (4) from (43).

Remark 2.9 (nonuniqueness and the change of energy H_0). Consider the initial two peakons $p_1\delta(x-x_1(0))+p_2\delta(x-x_2(0))$ with $x_1(0)<x_2(0)$ and $0<p_2<p_1$. Due to (4), the evolution system before collision for $x_1(t)$ and $x_2(t)$ is given by

$$\begin{cases}\frac{d}{dt}x_1(t)=\frac{1}{6}p_1^2+\frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)},\\ \frac{d}{dt}x_2(t)=\frac{1}{6}p_2^2+\frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)}.\end{cases} \tag{44}$$

Hence, they will collide at finite time $T_*=\frac{6(x_2(0)-x_1(0))}{p_1^2-p_2^2}$. When $t>T_*$, if we assume the two peakons stick together, according to (43) the evolution equation is given by

$$\frac{d}{dt}x_i(t)=\frac{1}{6}(p_1+p_2)^2,\ t>T_*,\quad i=1,2. \tag{45}$$

For $i=1,2$, we define

$$\hat{x}_i(t)=\begin{cases}x_i(t) \text{ given by (44) for } t<T_*,\\ x_i(t) \text{ given by (45) for } t>T_*,\end{cases} \tag{46}$$

and the sticky peakon weak solution

$$\hat{u}(x,t)=p_1G(x-\hat{x}_1(t))+p_2G(x-\hat{x}_2(t)),\quad \hat{m}=\hat{u}-\hat{u}_{xx}. \tag{47}$$

In this case, the energy H_0 (defined by (15)) of this sticky solution $\hat{m}$ is given by

$$H_0(\hat{m}(t))=\begin{cases}\frac{1}{2}(p_1^2+p_2^2)+p_1p_2e^{\hat{x}_1(t)-\hat{x}_2(t)}, & t<T_*,\\ \frac{1}{2}(p_1+p_2)^2, & t>T_*.\end{cases} \tag{48}$$

The energy H_0 is increasing before T_* and H_0 is continuous at the collision time T_*.

If we assume the two peakons cross each other after $t>T_*$ (still with amplitudes p_1, p_2), then according to (43), the evolution equations for $x_1(t)$ and $x_2(t)$ are given by

$$\begin{cases}\frac{d}{dt}x_1(t)=\frac{1}{6}p_1^2+\frac{1}{2}p_1p_2e^{x_2(t)-x_1(t)}, & t>T_*,\\ \frac{d}{dt}x_2(t)=\frac{1}{6}p_2^2+\frac{1}{2}p_1p_2e^{x_2(t)-x_1(t)}, & t>T_*.\end{cases} \tag{49}$$

This system is different with (4). For $i = 1, 2$, we define

$$\bar{x}_i(t) = \begin{cases} x_i(t) \ \text{ given by (44) for } \ t < T_*, \\ x_i(t) \ \text{ given by (49) for } \ t > T_*, \end{cases} \tag{50}$$

and the crossing peakon weak solution

$$\bar{u}(x,t) = p_1 G(x - \bar{x}_1(t)) + p_2 G(x - \bar{x}_2(t)), \quad \bar{m} = \bar{u} - \bar{u}_{xx}. \tag{51}$$

For the energy H_0 of the crossing solution $\bar{m}$, we have

$$H_0(\bar{m}(t)) = \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{-|\bar{x}_1(t) - \bar{x}_2(t)|} = \begin{cases} \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{\bar{x}_1(t) - \bar{x}_2(t)}, & t < T_*, \\ \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{\bar{x}_2(t) - \bar{x}_1(t)}, & t > T_*. \end{cases} \tag{52}$$

H_0 increases before time T_* and decreases after time T_*. H_0 is again continuous at the collision time T_*.

Both the sticky solution $u(x,t)$ and the crossing solution $\bar{u}(x,t)$ are two global peakon solutions, which proves nonuniqueness of weak solutions to the mCH equation. This nonuniqueness example can also be found in [12, Proposition 4.4].

The above example also shows that after collision, peakons can merge into one giving the sticky solution u, or cross each other yielding the crossing solution $\bar{u}$. Moreover, if we view T_* as the start point with one peakon, then the crossing solution $\bar{u}$ shows the scattering of one peakon. This indicates all the situation mentioned in question (iii) in Introduction.

At the end of this section, we give a useful proposition.

PROPOSITION 2.10. *Let $x_i(t)$, $1 \le i \le N$, be N Lipschitz functions in $[0,T)$ with $x_1(t) < x_2(t) < \cdots < x_N(t)$ and $p_1, \cdots, p_N$ are N non-zero constants. Then, $u^N(x,t) := \sum_{i=1}^N p_i G(x - x_i(t))$ is a weak solution to the mCH equation if and only if $x_i(t)$ satisfies (4).*

Proof. Obviously, we have

$$u^N \in C([0,T); H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R})).$$

In the following proof we denote $u := u^N$. For any test function $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$, let

$$\begin{aligned} \mathcal{L}(u,\phi) &= \int_0^T \int_{\mathbb{R}} u(\phi_t - \phi_{txx}) dxdt - \int_0^T \int_{\mathbb{R}} \left[\frac{1}{3}(u_x^3 \phi_{xx} + u^3 \phi_{xxx}) - (u^3 + u u_x^2)\phi_x \right] dxdt \\ &=: I_1 + I_2. \end{aligned} \tag{53}$$

Denote $x_0 := -\infty$, $x_{N+1} := +\infty$ and $p_0 = p_{N+1} = 0$. By integration by parts for space variable x, we calculate I_1 as

$$I_1 = \int_0^T \int_{\mathbb{R}} u(\phi_t - \phi_{txx})dxdt = \sum_{i=0}^N \int_0^T \int_{x_i}^{x_{i+1}} u(\phi_t - \phi_{txx})dxdt$$

$$= \sum_{i=0}^N \int_0^T \int_{x_i}^{x_{i+1}} \left(\frac{1}{2}\sum_{j\le i} p_j e^{x_j - x} + \frac{1}{2}\sum_{j>i} p_j e^{x-x_j} \right) (\phi_t - \phi_{txx})dxdt$$

$$= \int_0^T \sum_{i=1}^N p_i \phi_t(x_i(t), t)dt. \tag{54}$$

Similarly, for I_2 we have

$$I_2 = -\int_0^T \int_{\mathbb{R}} \left[\frac{1}{3}(u_x^3\phi_{xx} + u^3\phi_{xxx}) - (u^3 + uu_x^2)\phi_x \right] dxdt$$

$$= \int_0^T \sum_{i=1}^N p_i\phi_x(x_i(t)) \left(\frac{1}{6}p_i^2 + \frac{1}{2}\sum_{j<i} p_ip_je^{x_j - x_i} + \frac{1}{2}\sum_{j>i} p_ip_je^{x_i - x_j} \right.$$

$$\left. + \sum_{1\le m<i<n\le N} p_mp_ne^{x_m - x_n} \right) dt$$

$$= \int_0^T \sum_{i=1}^N p_i\phi_x(x_i(t))F(t)dt. \tag{55}$$

where

$$F(t) := \frac{1}{6}p_i^2 + \frac{1}{2}\sum_{j<i} p_ip_je^{x_j - x_i} + \frac{1}{2}\sum_{j>i} p_ip_je^{x_i - x_j} + \sum_{1\le m<i<n\le N} p_mp_ne^{x_m - x_n}.$$

Combining (53), (54) and (55) gives

$$\mathcal{L}(u,\phi) = \sum_{i=1}^N p_i \int_0^T \frac{d}{dt}\phi(x_i(t), t)dt + \int_0^T \sum_{i=1}^N p_i\phi_x(x_i(t)) \left(F(t) - \frac{d}{dt}x_i(t) \right) dt$$

$$= -\int_{\mathbb{R}} \phi(x, 0)dm_0^N + \int_0^T \sum_{i=1}^N p_i\phi_x(x_i(t)) \left(F(t) - \frac{d}{dt}x_i(t) \right) dt. \tag{56}$$

□

By Definition 2.2 we know u^N is a weak solution if and only if $\frac{d}{dt}x_i(t) = F(t)$, which is (4).

Remark 2.11. Proposition 2.10 implies the uniqueness of the limiting trajectories $x_i(t)$ before collisions. Consider the two peakon case in Remark 2.9. From Proposition 2.10, we know that solutions to (4) can not be used to construct peakon weak solutions after $t > T_*$. If we assume $x_1(t) > x_2(t)$ when $t > T_*$, Proposition 2.10 tells that (49) is the right evolution equation for $x_i(t)$, $i = 1, 2$.

3. Limiting peakon solutions as $\epsilon \to 0$. In this section, we analyze peakon solutions given by the dispersive regularization.

3.1. No collisions for the regularized system. In this subsection, we show that trajectories $\{x_i^\epsilon(t)\}_{i=1}^N$ obtained by (25) will never collide. Define

$$f_1^\epsilon(x) := \frac{1}{2}\int_0^\infty \rho_\epsilon(x-y)e^{-y}dy \quad \text{and} \quad f_2^\epsilon(x) := \frac{1}{2}\int_{-\infty}^0 \rho_\epsilon(x-y)e^{y}dy. \tag{57}$$

Changing variable gives

$$f_1^\epsilon(x) = \frac{1}{2}\int_{-\infty}^x \rho_\epsilon(y)e^{y-x}dy \quad \text{and} \quad f_2^\epsilon(x) = \frac{1}{2}\int_x^\infty \rho_\epsilon(y)e^{x-y}dy. \tag{58}$$

It is easy to see that both $f_1^\epsilon, f_2^\epsilon \in C^\infty(\mathbb{R})$ and we have the following lemma.

LEMMA 3.1. *Let $C_0 := ||\rho||_{L^\infty}$. Then, the following properties for f_i^ϵ $(i = 1, 2)$ hold:*

(i)

$$f_2^\epsilon(x) = f_1^\epsilon(-x), \quad G^\epsilon(x) = f_1^\epsilon + f_2^\epsilon, \quad \textit{and} \quad G_x^\epsilon(x) = -f_1^\epsilon(x) + f_2^\epsilon(x). \tag{59}$$

(ii)

$$||f_1^\epsilon||_{L^\infty}, ||f_2^\epsilon||_{L^\infty} \le \frac{1}{2}, \quad \textit{and} \quad ||\partial_x f_1^\epsilon||_{L^\infty}, ||\partial_x f_2^\epsilon||_{L^\infty} \le \frac{C_0}{2\epsilon} + \frac{1}{2}. \tag{60}$$

Proof. (i). The first two equalities in (59) can be easily proved. For the third one, taking derivative of (58) gives

$$\partial_x f_1^\epsilon(x) = \frac{1}{2}\rho_\epsilon(x) - f_1^\epsilon(x), \quad \text{and} \quad \partial_x f_2^\epsilon(x) = -\frac{1}{2}\rho_\epsilon(x) + f_2^\epsilon(x). \tag{61}$$ □

Hence, we have $G_x^\epsilon(x) = -f_1^\epsilon(x) + f_2^\epsilon(x)$.

(ii). By Definition (57), we can obtain

$$||f_1^\epsilon||_{L^\infty}, ||f_2^\epsilon||_{L^\infty} \le \frac{1}{2}.$$

Due to (61) and $C_0 = ||\rho||_{L^\infty}$, we have

$$||\partial_x f_1^\epsilon||_{L^\infty}, ||\partial_x f_2^\epsilon||_{L^\infty} \le \frac{C_0}{2\epsilon} + \frac{1}{2}.$$

THEOREM 3.2. *Let $\{x_i^\epsilon(t)\}_{i=1}^N$ be a solution to (25) subject to $x_i^\epsilon(0) = c_i$, $i = 1, \dots, N$ and $\sum_{i=1}^N |p_i| \le M_0$ for some constant M_0. If $c_1 < c_2 < \cdots < c_N$, then $x_1^\epsilon(t) < x_2^\epsilon(t) < \cdots < x_N^\epsilon(t)$ for all $t > 0$.*

Proof. If collisions between $\{x_i^\epsilon\}_{i=1}^N$ happen, we assume that the first collision is between x_k^ϵ and x_{k+1}^ϵ for some $1 \le k \le N-1$ at time $T_* > 0$. Our target is to prove $T_* = +\infty$.

By (21) and (59), we have

$$u^{N,\epsilon}(x,t) = \sum_{i=1}^N p_i G^\epsilon(x - x_i^\epsilon) = \sum_{i=1}^N p_i \left(f_1^\epsilon(x - x_i^\epsilon) + f_2^\epsilon(x - x_i^\epsilon)\right),$$

and

$$u_x^{N,\epsilon}(x,t) = \sum_{i=1}^N p_i G_x^\epsilon(x - x_i^\epsilon) = \sum_{i=1}^N p_i \left(-f_1^\epsilon(x - x_i^\epsilon) + f_2^\epsilon(x - x_i^\epsilon)\right).$$

Hence, we obtain

$$U_\epsilon^N(x,t) = (u^{N,\epsilon} + u_x^{N,\epsilon})(u^{N,\epsilon} - u_x^{N,\epsilon}) = 4\left(\sum_{i=1}^N p_i f_2^\epsilon(x - x_i^\epsilon)\right)\left(\sum_{i=1}^N p_i f_1^\epsilon(x - x_i^\epsilon)\right).$$

From (25), we have

$$\frac{d}{dt}x_k^\epsilon = \left[\rho_\epsilon * U_\epsilon^N\right](x_k^\epsilon) \quad \text{and} \quad \frac{d}{dt}x_{k+1}^\epsilon = \left[\rho_\epsilon * U_\epsilon^N\right](x_{k+1}^\epsilon). \tag{62}$$

For $t < T_*$, taking the difference gives

$$\begin{aligned}
&\frac{d}{dt}(x_{k+1}^\epsilon - x_k^\epsilon)\\
=&4\int_{\mathbb{R}} \rho_\epsilon(y)\left(\sum_{i=1}^N p_i f_2^\epsilon(x_{k+1}^\epsilon - y - x_i^\epsilon)\right)\left(\sum_{i=1}^N p_i f_1^\epsilon(x_{k+1}^\epsilon - y - x_i^\epsilon)\right)dy\\
&-4\int_{\mathbb{R}} \rho_\epsilon(y)\left(\sum_{i=1}^N p_i f_2^\epsilon(x_k^\epsilon - y - x_i^\epsilon)\right)\left(\sum_{i=1}^N p_i f_1^\epsilon(x_k^\epsilon - y - x_i^\epsilon)\right)dy\\
=&4\int_{\mathbb{R}} \rho_\epsilon(y)\left(\sum_{i=1}^N p_i f_2^\epsilon(x_{k+1}^\epsilon - y - x_i^\epsilon)\right)\sum_{i=1}^N p_i\left(f_1^\epsilon(x_{k+1}^\epsilon - y - x_i^\epsilon) - f_1^\epsilon(x_k^\epsilon - y - x_i^\epsilon)\right)dy\\
&+4\int_{\mathbb{R}} \rho_\epsilon(y)\left(\sum_{i=1}^N p_i f_1^\epsilon(x_k^\epsilon - y - x_i^\epsilon)\right)\sum_{i=1}^N p_i\left(f_2^\epsilon(x_{k+1}^\epsilon - y - x_i^\epsilon) - f_2^\epsilon(x_k^\epsilon - y - x_i^\epsilon)\right)dy.
\end{aligned}$$

Combining (59) and (60) yields

$$\begin{aligned}
\left|\frac{d}{dt}(x_{k+1}^\epsilon - x_k^\epsilon)\right| \le& 2M_0^2||\partial_x f_1^\epsilon||_{L^\infty}(x_{k+1}^\epsilon - x_k^\epsilon) + 2M_0^2||\partial_x f_2^\epsilon||_{L^\infty}(x_{k+1}^\epsilon - x_k^\epsilon)\\
\le& C_\epsilon(x_{k+1}^\epsilon - x_k^\epsilon), \quad t < T_*,
\end{aligned} \tag{63}$$

where

$$C_\epsilon = M_0^2\left(\frac{C_0}{\epsilon} + 1\right).$$

Hence, for $t < T_*$ we have

$$-C_\epsilon(x_{k+1}^\epsilon - x_k^\epsilon) \le \frac{d}{dt}(x_{k+1}^\epsilon - x_k^\epsilon) \le C_\epsilon(x_{k+1}^\epsilon - x_k^\epsilon), \tag{64}$$

∎

which implies

$$0 < (c_{k+1} - c_k)e^{-C_\epsilon t} \le x_{k+1}^\epsilon(t) - x_k^\epsilon(t) \quad \text{for} \quad t < T_*.$$

By our assumption about T_*, we know $T_* = +\infty$. Hence, we have $x_1^\epsilon(t) < x_2^\epsilon(t) < \cdots < x_N^\epsilon(t)$ for all $t > 0$.

Remark 3.3. Let $u^N(x,t) = \sum_{i=1}^N G(x - x_i(t))$ be an N-peakon solution to the mCH equation obtained by Theorem 2.4. From Theorem 3.2, we have

$$x_1(t) \le x_2(t) \le \cdots \le x_N(t). \tag{65}$$

This result shows that the limit solution allows no crossing between peakons.

3.2. Two peakon solutions. As mentioned in Introduction, the sticky peakon solutions given in [12] also satisfy (65). In this subsection, when $N = 2$, we show that the limiting N-peakon solutions given in Theorem 2.4 agree with sticky peakon solutions (see $u(x,t)$ in Remark 2.9). Due to Proposition 2.10, the cases with no collisions are easy to verify.

Consider the case with a collision for $N = 2$. When $p_1^2 > p_2^2$ and $x_1(0) = c_1 < c_2 = x_2(0)$, the equations for $x_1(t)$ and $x_2(t)$ before collisions are given by

$$\begin{cases} \frac{d}{dt}x_1(t) = \frac{1}{6}p_1^2 + \frac{1}{2}e^{x_1(t)-x_2(t)}, \\ \frac{d}{dt}x_2(t) = \frac{1}{6}p_2^2 + \frac{1}{2}e^{x_1(t)-x_2(t)}. \end{cases} \tag{66}$$

The two peakons collide at $T_* = \frac{6(c_2-c_1)}{p_1^2-p_2^2}$. Next, we prove the following theorem.

THEOREM 3.4. *Assume $N = 2$ and $m_0^N(x) = p_1\delta(x-c_1)+p_2\delta(x-c_2)$ with $p_1^2 > p_2^2$ and $c_1 < c_2$. Then, the peakon solution $u^N(x,t) = p_1G(x-x_1(t)) + p_2G(x-x_2(t))$ obtained in Theorem 2.4 is a sticky peakon solution, which means*

$$x_1(t) = x_2(t) \quad for \quad t \geq T_* := \frac{6(c_2-c_1)}{p_1^2-p_2^2}. \tag{67}$$

To prove Theorem 3.4, we first consider (25) for $N = 2$. Denote $S_\epsilon(t) := x_2^\epsilon(t) - x_1^\epsilon(t) > 0$. By the fact that $f_1^\epsilon(-x) = f_2^\epsilon(x)$, we find that

$$\begin{aligned} \frac{d}{dt}x_1^\epsilon &= 4\int_{-\infty}^{\infty}\rho_\epsilon(y)\big[p_1f_2^\epsilon(-y) + p_2f_2(-S_\epsilon - y)\big]\big[p_1f_1^\epsilon(-y) + p_2f_1^\epsilon(-S_\epsilon - y)\big]dy \\ &= 4\int_{-\infty}^{\infty}\rho_\epsilon(y)\big[p_1f_1^\epsilon(y) + p_2f_1^\epsilon(S_\epsilon + y)\big]\big[p_1f_2^\epsilon(y) + p_2f_2^\epsilon(S_\epsilon + y)\big]dy. \end{aligned} \tag{68}$$

By changing of variables $y \to -y$ and using the fact that ρ_ϵ is even, we obtain

$$\begin{aligned} \frac{d}{dt}x_2^\epsilon &= 4\int_{-\infty}^{\infty}\rho_\epsilon(y)\big[p_1f_2^\epsilon(S_\epsilon - y) + p_2f_2(-y)\big]\big[p_1f_1^\epsilon(S_\epsilon - y) + p_2f_1^\epsilon(-y)\big]dy \\ &= 4\int_{-\infty}^{\infty}\rho_\epsilon(y)\big[p_1f_2^\epsilon(S_\epsilon + y) + p_2f_2^\epsilon(y)\big]\big[p_1f_1^\epsilon(S_\epsilon + y) + p_2f_1^\epsilon(y)\big]dy \end{aligned} \tag{69}$$

Taking the difference of (68) and (69) gives

$$\frac{d}{dt}S_\epsilon = 4(p_2^2 - p_1^2)\int_{-\infty}^{\infty}\rho_\epsilon(y)\big[f_1^\epsilon(y)f_2^\epsilon(y) - f_1^\epsilon(S_\epsilon + y)f_2^\epsilon(S_\epsilon + y)\big]dy. \tag{70}$$

We have the following useful proposition, the proof of which is in Appendix.

PROPOSITION 3.5. *For any $s > 0$, we have*

$$\lim_{\epsilon\to 0} 4\int_{-\infty}^{\infty}\rho_\epsilon(x)\big[f_1^\epsilon(x)f_2^\epsilon(x) - f_1^\epsilon(s+x)f_2^\epsilon(s+x)\big]dx = \frac{1}{6}. \tag{71}$$

The above convergence is uniform about $s \in [\delta, +\infty)$ for any $\delta > 0$.

Proof of Theorem 3.4. Let $m_0^N(x) = p_1\delta(x-c_1) + p_2\delta(x-c_2)$ for constants p_i and c_i satisfying

$$c_1 < c_2 \quad \text{and} \quad p_1^2 > p_2^2. \tag{72}$$

$x_1^\epsilon(t)$ and $x_2^\epsilon(t)$ are obtained by (25). From Theorem 3.1, we have $x_1^\epsilon(t) < x_2^\epsilon(t)$ for any $t \ge 0$. By Theorem 2.4, for any $T > 0$, there are $x_1(t), x_2(t) \in C([0,T])$ such that

$$x_1^\epsilon(t) \to x_1(t) \;\text{ and }\; x_2^\epsilon(t) \to x_2(t) \;\text{ in }\; C([0,T]), \quad \epsilon \to 0.$$

Hence, we have

$$x_1(t) \le x_2(t).$$

By Proposition 2.10, we know that solution given by Theorem 2.4 is the same as the sticky peakon solution when $t < T_*$.

By (70) and Proposition 3.5, we can see that for any $0 < \delta < \min\left\{c_2 - c_1, -\frac{1}{6}(p_2^2 - p_1^2)\right\}$, there is a $\epsilon_0 > 0$ such that when $S_\epsilon(t) \ge \delta$ we have

$$\frac{1}{6}(p_2^2 - p_1^2) - \delta < \frac{d}{dt}S_\epsilon(t) < \frac{1}{6}(p_2^2 - p_1^2) + \delta < 0 \;\text{ for any }\; \epsilon < \epsilon_0.$$

Claim 1: If there exists $t_0 > 0$ such that $S_\epsilon(t_0) \le \delta$, then $S_\epsilon(t) \le \delta$ for $t > t_0$. Indeed, if there is $t_1 > t_0$ and $S_\epsilon(t_1) > \delta$, we set

$$t_2 := \inf\{t < t_1 : S_\epsilon(s) > \delta \text{ for } s \in (t, t_1)\}.$$

Hence, $t_2 \ge t_0$ and $S_\epsilon(t_2) = \delta$. Moreover, $S_\epsilon(t) > \delta$ for $t \in (t_2, t_1)$. Therefore,

$$S_\epsilon(t_1) = \int_{t_2}^{t_1} \frac{d}{ds}S_\epsilon(s)ds + S_\epsilon(t_2) \le \left[\frac{1}{6}(p_2^2 - p_1^2) + \delta\right](t_1 - t_2) + \delta \le \delta,$$

which is a contradiction with $S_\epsilon(t_1) > \delta$.

Claim 2: We have $S_\epsilon(t) \le \delta$ for $t \ge \frac{6(c_2 - c_1 - \delta)}{p_1^2 - p_2^2 - 6\delta} =: t_\delta$. If not, from Claim 1 we have $S_\epsilon(t) > \delta$ for $t \le t_\delta$. Hence,

$$S_\epsilon(t_\delta) = \int_0^{t_\delta} \frac{d}{ds}S_\epsilon(s)ds + c_2 - c_1 \le \left[\frac{1}{6}(p_2^2 - p_1^2) + \delta\right]t_\delta + c_2 - c_1 \le \delta,$$

which is a contradiction.

With the above claims, we can obtain

$$\lim_{\epsilon \to 0} S_\epsilon(t) = 0 \;\text{ for }\; t \ge \frac{6(c_2 - c_1)}{p_1^2 - p_2^2}, \tag{73}$$

∎

which implies (67)

Remark 3.6. Though the peakons are not physical particles and they are not governed by Newton's laws, we have the analogy of the conservation of momentum during the collision. Let p be the 'mass' of the peakon. The speeds of the two peakons before collision are $\frac{1}{6}p_1^2 + \frac{1}{2}p_1p_2$ and $\frac{1}{6}p_2^2 + \frac{1}{2}p_1p_2$ respectively. The speed after collision is $\frac{1}{6}(p_1 + p_2)^2$. We can check formally that

$$(p_1 + p_2)\frac{1}{6}(p_1 + p_2)^2 = p_1\left(\frac{1}{6}p_1^2 + \frac{1}{2}p_1p_2\right) + p_2\left(\frac{1}{6}p_2^2 + \frac{1}{2}p_1p_2\right).$$

We can then introduce the instantaneous (infinite) "force" as

$$F_1 = p_1[\dot{x}_1]\delta(t - T_*) = \frac{1}{6}p_1p_2(p_2 - p_1)\delta(t - T_*),$$

where $[\dot{x}_1]$ represents the jump of $\dot{x}$ at $t = T_*$. Similarly,

$$F_2 = p_2[\dot{x}_2]\delta(t - T_*) = \frac{1}{6}p_2p_1(p_1 - p_2)\delta(t - T_*).$$

Here $F_1 + F_2 = 0$, which is equivalent to the "local conservation of momentum".

3.3. Discussion about three particle system. When $N \geq 3$, the limiting N-peakon solutions obtained by Theorem 2.4 can be complicated. In this subsection, we study the interactions between three peakon trajectories.

Denote the initial data $x_1(0) < x_2(0) < x_3(0)$ and constant amplitudes of peakons $p_i > 0$, $i = 1, 2, 3$. Let $x_i^\epsilon(t)$, $i = 1, 2, 3$, be solutions to the regularized system (25) and $x_i(t)$, $i = 1, 2, 3$, be the limiting trajectories given by Theorem 2.4. Let $x_i^s(t)$, $i = 1, 2, 3$, be trajectories to sticky peakon solutions given in [12]. Before the first collision time, by Proposition 2.10 we know that $x_i(t) = x_i^s(t)$, $i = 1, 2, 3$, which is the solution to (4). However, after collisions, the limiting trajectories $x_i(t)$ may or may not coincide with the sticky trajectories $x_i^s(t)$. Below, we consider two typical cases.

Sticky case (i). We illustrate this case by an example with $p_1 = 4,\ p_2 = 2,\ p_3 = 1$ and $x_1(0) = -7,\ x_2(0) = -5,\ x_3(0) = -3$ (see Figure 1). For the sticky trajectories (red dashed lines in Figure 1) $x_i^s(t)$, $i = 1, 2, 3$, the first collision happens between $x_2^s(t)$ and $x_3^s(t)$ at time t_1^*. Then $x_2^s(t)$ and $x_3^s(t)$ sticky together traveling with new amplitude $p_2 + p_3$ for $t \in (t_1^*, t_2^*)$. Because $p_1 > p_2 + p_3$, $x_1^s(t)$ catches up with $x_2^s(t)$ and $x_3^s(t)$ at t_2^*. At last, the three peakons all sticky together after t_2^*.

When $\epsilon > 0$ is small, the behavior of trajectories $x_i^\epsilon(t)$, $i = 1, 2, 3$, given by the regularized system (25) is very similar to the sticky trajectories (see blue solid lines in Figure 1). This indicates that $x_i(t) \equiv x_i^s(t)$ for any $t > 0$ and the limiting peakon solution given by Theorem 2.4 agrees with the sticky peakon solution.

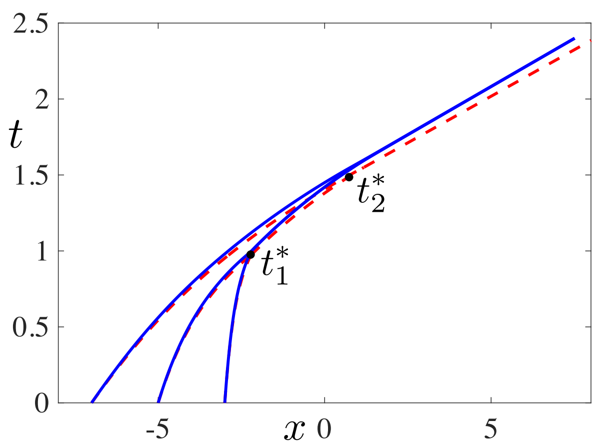

FIG. 1. $p_1 = 4,\ p_2 = 2,\ p_3 = 1$ *and* $x_1(0) = -7,\ x_2(0) = -5,\ x_3(0) = -3$; $\epsilon = 0.02$. *The blue lines are trajectories of three peakons* $\{x_i^\epsilon(t)\}_{i=1}^3$ *given by dispersive regularization system* (25). *The red dashed lines are trajectories of sticky three peakons.*

Sticky and separation case (ii). We illustrate this case by an example with $p_1 = 4,\ p_2 = 2,\ p_3 = 3$ and $x_1(0) = -7,\ x_2(0) = -6,\ x_3(0) = -2$ (see Figure 2). For the sticky trajectories (red dashed lines in Figure 2) $x_i^s(t)$, $i = 1, 2, 3$, the first collision happens between $x_1^s(t)$ and $x_2^s(t)$ at time $\hat{t}_1$. Then $x_1^s(t)$ and $x_2^s(t)$ sticky together traveling with new amplitude $p_1 + p_2$ for $t \in (\hat{t}_1, \hat{t}_2)$. Because $p_1 + p_2 > p_3$, $x_1^s(t)$ and $x_2^s(t)$ catch up with $x_3^s(t)$ at $\hat{t}_2$. At last, the three peakons all sticky together after $\hat{t}_2$.

When $\epsilon > 0$ is small, the behavior of trajectories $x_i^\epsilon(t)$, $i = 1, 2, 3$, given by the regularized system (25) is very similar with the sticky trajectories $x_i^s(t)$ before T_1, where $x_1^\epsilon(t)$ get close to $x_2^\epsilon(t)$. However, when $x_3^\epsilon(t)$ comes close to $x_2^\epsilon(t)$, $x_2^\epsilon(t)$ separates from $x_1^\epsilon(t)$ around T_1 and gradually moves to $x_3^\epsilon(t)$ and then holds together with $x_3^\epsilon(t)$. Since $p_2+p_3>p_1$, $x_2^\epsilon(t)$ and $x_3^\epsilon(t)$ get far away from $x_1^\epsilon(t)$.

This indicates the limiting trajectories $x_i(t)\neq x_i^s(t)$ for $t\geq T_1$ and the limiting peakon solution given by Theorem 2.4 does not agree with the sticky peakon solution. Below, we give some discussions about this interesting phenomenon.

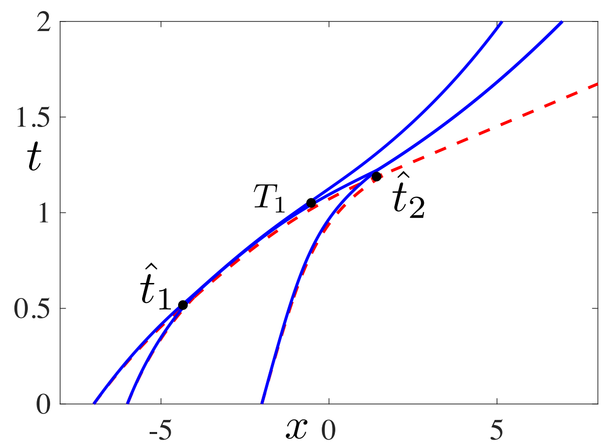

FIG. 2. $p_1=4,\ p_2=2,\ p_3=3$ *and* $x_1(0)=-7,\ x_2(0)=-6,\ x_3(0)=-2$; $\epsilon=0.02$. *The blue lines are trajectories for three peakons* $\{x_i^\epsilon(t)\}_{i=1}^3$ *obtained by dispersive regularization system* (25). *The red dashed lines are trajectories of sticky three peakons.*

Next, we discuss in detail the limiting solution in cases like Figure 2, i.e. $p_1>p_2>0$, $p_1+p_2>p_3>0$, $p_1<p_2+p_3$ and $x_3(0)-x_2(0)\gg x_2(0)-x_1(0)>0$. Consider the limiting solution of the form:

$$u(x,t)=\sum_{i=1}^{3}p_iG(x-x_i(t)),$$

where $x_i(t)$ are Lipschitz continuous and $x_1(t)\leq x_2(t)\leq x_3(t)$. Since $x_1(0)<x_2(0)<x_3(0)$, by Proposition 2.10, $x_i(t): i=1,2,3$ satisfy the following system for $t\in(0,T_*)$ where $T_*>0$ is the first collision time:

$$\begin{cases}\dfrac{dx_1}{dt}=\dfrac{1}{6}p_1^2+\dfrac{1}{2}p_1p_2e^{-(x_2-x_1)}+\dfrac{1}{2}p_1p_3e^{-(x_3-x_1)},\\ \dfrac{dx_2}{dt}=\dfrac{1}{6}p_2^2+\dfrac{1}{2}p_1p_2e^{-(x_2-x_1)}+\dfrac{1}{2}p_2p_3e^{-(x_3-x_2)}+p_1p_3e^{-(x_3-x_1)},\\ \dfrac{dx_3}{dt}=\dfrac{1}{6}p_3^2+\dfrac{1}{2}p_1p_3e^{-(x_3-x_1)}+\dfrac{1}{2}p_2p_3e^{-(x_3-x_2)}.\end{cases}\tag{74}$$

Let $S_i:=x_{i+1}-x_i\geq 0$, $i=1,2$. From (74), the distances S_i satisfy the following equations for $t<T_*$:

$$\begin{cases}\dfrac{dS_1}{dt}=\dfrac{1}{6}(p_2^2-p_1^2)+\dfrac{1}{2}p_2p_3e^{-S_2}+\dfrac{1}{2}p_1p_3e^{-(S_1+S_2)},\\ \dfrac{dS_2}{dt}=\dfrac{1}{6}(p_3^2-p_2^2)-\dfrac{1}{2}p_1p_2e^{-S_1}-\dfrac{1}{2}p_1p_3e^{-(S_1+S_2)}.\end{cases}\tag{75}$$

For the case in Figure 2 to happen, $S_2(0)$ should be large enough so that $S_1(T_*)=0$ and

$$\lim_{t\to T_*^-}\frac{dS_1}{dt}=\frac{1}{6}(p_2^2-p_1^2)+\frac{1}{2}p_2p_3e^{-S_2(T_*)}+\frac{1}{2}p_1p_3e^{-S_2(T_*)}<0.$$

In other words, $S_2(T_*) > S_2^* > 0$, where S_2^* is defined by:

$$\frac{1}{6}(p_2^2 - p_1^2) + \frac{1}{2}p_2p_3e^{-S_2^*} + \frac{1}{2}p_1p_3e^{-S_2^*} = 0.$$

Since $S_1(t) \geq 0$, while

$$\frac{1}{6}(p_2^2 - p_1^2) + \frac{1}{2}p_2p_3e^{-S_2} + \frac{1}{2}p_1p_3e^{-(S_1+S_2)} < 0,$$

(75) must not be valid for $t \in (T_*, T_* + \delta)$ for some $\delta > 0$ and neither does (74). Indeed, the new system of equations must be (4) for $N = 2$:

$$(76) \qquad \begin{cases} \dfrac{d}{dt}x_i(t) = \dfrac{1}{6}(p_1+p_2)^2 + \dfrac{1}{2}(p_1+p_2)p_3e^{x_i(t)-x_3(t)},\ i = 1, 2, \\ \dfrac{d}{dt}x_3(t) = \dfrac{1}{6}p_3^2 + \dfrac{1}{2}(p_1+p_2)p_3e^{x_2(t)-x_3(t)}. \end{cases}$$

Hence, $S_1(t) = 0$ for $t \in (T_*, T_*+\delta)$ while $S_2(t)$ keeps decreasing because $p_1+p_2 > p_3$.

Note that the sticky solutions $x_i^s(t)$ satisfy (76) until $x_1^s(t) = x_2^s(t) = x_3^s(t)$. On the contrary, the simulations indicate that $x_1(t)$ and $x_2(t)$ can split when $x_2(t) < x_3(t)$ and then $\{x_i(t)\}_{i=1}^3$ do not satisfy (76) after the splitting. Define the splitting time T_1 as

$$T_1 = \inf\{t \geq T_* : S_1(t) > 0\}.$$

We claim that $T_1 \geq T_2 := \inf\{t > 0 : S_2(t) = S_2^*\} > T_*$. Suppose for otherwise $T_1 < T_2$, then there exists $\delta > 0$ such that $S_1(t) > 0$ for $t \in (T_1, T_1 + \delta)$ with some small δ, $S_1(T_1) = 0$ and $S := \inf_{t\in(T_1,T_1+\delta)} S_2(t) > S_2^*$. For $t \in (T_1, T_1 + \delta)$, S_1 and S_2 must satisfy (75) by Proposition 2.10. Consequently,

$$\frac{d}{dt}S_1(t) \leq \frac{1}{6}(p_2^2 - p_1^2) + \frac{1}{2}p_2p_3e^{-S} + \frac{1}{2}p_1p_3e^{-S} < 0,\ t \in (T_1, T_1 + \delta).$$

Since $S_1(T_1) = 0$, we must have $S_1(t) \leq 0$ for $t \in (T_1, T_1+\delta)$. This is a contradiction.

Now that (76) holds on (T_*, T_1) while $T_1 \geq T_2$, we find

$$T_2 = T_* + 6(S_2(T_*) - S_2^*)/((p_1 + p_2)^2 - p_3^2) > T_*.$$

The question is that when the split happens (i.e. how large can T_1 be).

Conjecture. *At the point of splitting ($t = T_1$), both $x_1(t)$ and $x_2(t)$ are right-differentiable, and $x_1(t) : t \geq T_1$ and $x_2(t) : t \geq T_1$ are tangent at $t = T_1$.*

If this conjecture is valid, then we must have

$$\lim_{t\to T_1^+} \frac{d}{dt}S_1(t) = 0$$

and therefore

$$T_1 = T_2.$$

In summary, the dispersive regularization limit weak solution is quite different from the sticky particle model in [12] when $N \geq 3$. Another difference we note is that the sticky particle model has bifurcation instability for the dynamics of three peakon system: consider a three particles system with initial data: $p_1 = 4, x_1(0) = -4$, $p_2 = 3, x_2(0) \in (-4, 4)$ and $p_3 = 2, x_3(0) = 4$. There exists $x_c \in (-4, 4)$ such that in the $x_2(0) > x_c$ cases, the second and third peakons merge first and then they move apart from the first one (see Figure 3 (b)), while $x_2(0) < x_c$ implies that the first two merge first and then they catch up with the third one, merging into a single particle (see Figure 3 (a)). This is a kind of bifurcation instability due to the initial position of the second peakon: a little change in $x_2(0)$ results in very different solutions at later time. It seems that the $\epsilon \to 0$ limit does not possess such instability due to the splitting as in Figure 2.

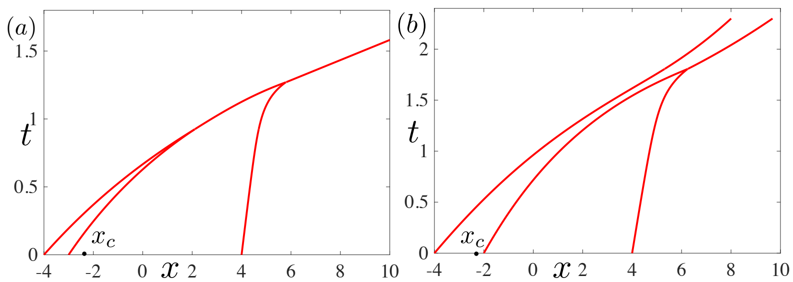

FIG. 3. *(a).* $p_1 = 4,\ p_2 = 3,\ p_3 = 2$ *and* $x_1(0) = -4,\ x_2(0) = -3,\ x_3(0) = 4$. *The three peakons merge into one peakon. (b).* $p_1 = 4,\ p_2 = 3,\ p_3 = 2$ *and* $x_1(0) = -4,\ x_2(0) = -2,\ x_3(0) = 4$. *The three peakons merge into two separated peakons.*

4. Mean field limit.

In this section, we use a particle method to prove global existence of weak solutions to the mCH equation for general initial data $m_0 \in \mathcal{M}(\mathbb{R})$.

Assume that the initial date m_0 satisfies

$$m_0 \in \mathcal{M}(\mathbb{R}), \quad \operatorname{supp}\{m_0\} \subset (-L, L), \quad M_0 := \int_{\mathbb{R}} d|m_0| < +\infty. \tag{77}$$

Let us choose the initial data $\{c_i\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ to approximate $m_0(x)$. Divide the interval $[-L, L]$ into N non-overlapping sub-interval I_j by using the uniform grid with size $h = \frac{2L}{N}$. We choose c_i and p_i as

$$c_i := -L + (i - \frac{1}{2})h; \quad p_i := \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} dm_0, \quad i = 1, 2, \cdots, N. \tag{78}$$

Hence, we have

$$\sum_{i=1}^{N} |p_i| \le \int_{[-L,L]} d|m_0| \le M_0. \tag{79}$$

Using (78), one can easily prove that m_0 is approximated by

$$m_0^N(x) := \sum_{j=1}^{N} p_j \delta(x - c_j) \tag{80}$$

in the sense of measures. Actually, for any test function $\phi \in C_b(\mathbb{R})$, we know ϕ is uniformly continuous on $[-L, L]$. Hence, for any $\eta > 0$, there exists a $\delta > 0$ such that when $x, y \in [-L, L]$ and $|x - y| < \delta$, we have $|\phi(x) - \phi(y)| < \eta$. Hence, choose $\frac{h}{2} < \delta$

and we have

$$\left| \int_{\mathbb{R}} \phi(x) dm_0 - \int_{\mathbb{R}} \phi(x) dm_0^N \right| = \left| \int_{[-L,L]} \phi(x) dm_0 - \int_{[-L,L]} \phi(x) dm_0^N \right|$$

$$= \left| \sum_{i=1}^{N} \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} \big(\phi(x) - \phi(c_i)\big) dm_0 \right| \le \eta \sum_{i=1}^{N} \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} d|m_0| \le M_0 \eta. \tag{81}$$

Let $\eta \to 0$ and we obtain the narrow convergence from $m_0^N(x)$ to $m_0(x)$.

For initial data $m_0^N(x)$, Theorem 2.4 gives a weak solution $u^N(x,t) = \sum_{i=1}^N p_i G(x - x_i(t))$, where $x_i(0) = c_i$ and p_i are given by (78). Moreover, (31) holds for $x_i(t)$, $1 \le i \le N$.

Next, we are going to use some space-time BV estimates to show compactness of u^N. To this end, we recall the definition of BV functions.

Definition 4.1. (i). *For dimension $d \ge 1$ and an open set $\Omega \subset \mathbb{R}^d$, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if*

$$Tot.Var.\{f\} := \sup \left\{ \int_{\Omega} f(x) \nabla \cdot \phi(x) dx : \phi \in C_c^1(\Omega; \mathbb{R}^d), \quad ||\phi||_{L^\infty} \le 1 \right\} < \infty.$$

(ii). *(Equivalent definition for one dimension case) A function f belongs to $BV(\mathbb{R})$ if for any $\{x_i\} \subset \mathbb{R}$, $x_i < x_{i+1}$, the following statement holds:*

$$Tot.Var.\{f\} := \sup_{\{x_i\}} \left\{ \sum_i |f(x_i) - f(x_{i-1})| \right\} < \infty.$$

Remark 4.2. Let $\Omega \subset \mathbb{R}^d$ for $d \ge 1$ and $f \in BV(\Omega)$. $Df := (D_{x_1} f, \dots, D_{x_d} f)$ is the distributional gradient of f. Then, Df is a vector Radon measure and the total variation of f is equal to the total variation of $|Df|$: $Tot.Var.\{f\} = |Df|(\Omega)$. Here, $|Df|$ is the total variation measure of the vector measure Df ([20, Definition (13.2)]).

If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Definition 4.1 (ii), then f satisfies Definition (i). On the contrary, if f satisfies Definition 4.1 (i), then there exists a right continuous representative which satisfies Definition (ii). See [20, Theorem 7.2] for the proof.

Now, we give some space and time BV estimates about $u^N, \partial_x u^N$, which is similar to [12, Proposition 3.3].

Proposition 4.3. *Assume initial value m_0 satisfies (77). p_i and c_i, $1 \le i \le N$, are given by (78) and m_0^N is defined by (80). Let $u^N(x,t) = \sum_{i=1}^N p_i G(x - x_i(t))$ be the N-peakon solution given by Theorem 2.4 subject to initial data $m^N(x,0) = (1 - \partial_{xx}) u^N(x,0) = m_0^N(x)$. Then, the following statements hold.*

(i). *For any $t \in [0, \infty)$, we have*

$$Tot.Var.\{u^N(\cdot,t)\} \le M_0, \quad Tot.Var.\{\partial_x u^N(\cdot,t)\} \le 2M_0 \text{ uniformly in } N. \tag{82}$$

(ii).

$$||u^N||_{L^\infty} \le \frac{1}{2} M_0, \quad ||\partial_x u^N||_{L^\infty} \le \frac{1}{2} M_0 \text{ uniformly in } N. \tag{83}$$

(iii). *For $t, s \in [0, \infty)$, we have*

$$\int_{\mathbb{R}} |u^N(x,t) - u^N(x,s)| dx \le \frac{1}{2} M_0^3 |t - s|, \quad \int_{\mathbb{R}} |\partial_x u^N(x,t) - \partial_x u^N(x,s)| dx \le M_0^3 |t - s|. \tag{84}$$

(iv). *For any $T > 0$, there exist subsequences of u^N, u_x^N (also labeled as u^N, u_x^N) and two functions u, $u_x \in BV(\mathbb{R}\times[0,T))$ such that*

$$u^N \to u, \quad u_x^N \to u_x \ \text{in}\ L^1_{loc}(\mathbb{R}\times[0,+\infty)) \ \text{as}\ N\to\infty, \tag{85}$$

and u, u_x satisfy all the properties in (i), (ii) *and* (iii).

Proof. See [12, Proposition 3.3]. We remark that the key estimate to prove (84) is (31). ∎

With Proposition 4.3, we have the following theorem:

THEOREM 4.4. *Let the assumptions in Proposition 4.3 hold. Then, the following statements hold:*

(i). *The limiting function u obtained in Proposition 4.3* ((iv)) *satisfies*

$$u \in C([0,+\infty); H^1(\mathbb{R})) \cap L^\infty(0,+\infty; W^{1,\infty}(\mathbb{R})) \tag{86}$$

and it is a global weak solution of the mCH equation (1).

(ii). *For any $T > 0$, we have*

$$m = (1-\partial_{xx})u \in \mathcal{M}(\mathbb{R}\times[0,T))$$

and there exists a subsequence of m^N (also labeled as m^N) such that

$$m^N \stackrel{*}{\rightharpoonup} m \ \text{in}\ \mathcal{M}(\mathbb{R}\times[0,T)) \quad (\text{as}\ N\to+\infty). \tag{87}$$

(iii). *For a.e. $t \ge 0$ we have (in subsequence sense)*

$$m^N(\cdot,t) \stackrel{*}{\rightharpoonup} m(\cdot,t) \ \text{in}\ \mathcal{M}(\mathbb{R}) \ \text{as} \ \ N\to+\infty \tag{88}$$

and

$$\text{supp}\{m(\cdot,t)\} \subset \Big(-L-\frac{1}{2}M_0^2t, L+\frac{1}{2}M_0^2t\Big), \tag{89}$$

Proof. The proof is similar to [12, Theorem 3.4] and we omit it. ∎

Remark 4.5. We remark that when m_0 is a positive Radon measure, m is also positive. Actually, $m_0 \in \mathcal{M}_+(\mathbb{R})$ implies that $p_i \ge 0$ and $m^{N,\epsilon} \ge 0$. Therefore, the limiting measure m belongs to $\mathcal{M}_+(\mathbb{R}\times[0,T))$. By the same methods as in [12, Theorem 3.5], we can also show that for a.e. $t \ge 0$,

$$m(\cdot,t)(\mathbb{R}) = m_0(\mathbb{R}), \quad |m(\cdot,t)|(\mathbb{R}) \le |m_0|(\mathbb{R}). \tag{90}$$

5. Modified equation and dispersive effects. Note that the regularization for the N-peakon solutions can be equivalently reformulated as the regularization performed directly on the equation. We consider the equation

$$m_t + \Big[m\Big(\rho_\epsilon * \big((\rho_\epsilon * u)^2 - (\rho_\epsilon * u_x)^2\big)\Big)\Big]_x = 0, \quad m = u - u_{xx}. \tag{91}$$

To see the equivalence, consider its characteristic equation

$$\begin{cases} \dot{X}(\xi,t) = \rho_\epsilon * \big((\rho_\epsilon * u)^2 - (\rho_\epsilon * u_x)^2\big)(X(\xi,t),t), \\ X(\xi,0) = \xi \in \mathbb{R}. \end{cases} \tag{92}$$

Due to the relation between u and m, we have

$$(\rho_\epsilon * u)(x) = \int_{\mathbb{R}} \rho_\epsilon(x-y) \int_{\mathbb{R}} G(y-z)m(z)dzdy = \int_{\mathbb{R}} G^\epsilon(x-z)m(z)dz = \int_{\mathbb{R}} G^\epsilon(x - X(\theta,t))m_0(\theta)d\theta. \tag{93}$$

We define

$$\begin{aligned} U_\epsilon(x,t) &:= (\rho_\epsilon * u)^2(x,t) - (\rho_\epsilon * u_x)^2(x,t) \\ &= \left(\int_{\mathbb{R}} G^\epsilon(x - X(\theta,t))m_0(\theta)d\theta\right)^2 - \left(\int_{\mathbb{R}} G^\epsilon_x(x - X(\theta,t))m_0(\theta)d\theta\right)^2, \end{aligned} \tag{94}$$

and

$$U^\epsilon(x,t) = [\rho_\epsilon * U_\epsilon](x,t).$$

Equation (92) can be rewritten as

$$\begin{cases} \dot{X}(\xi,t) = U^\epsilon(X(\xi,t),t), \\ X(\xi,0) = \xi \in \mathbb{R}. \end{cases} \tag{95}$$

Because the velocity field U^ϵ is bounded and smooth, one may show that Equation (95) has a global solution for given initial data $m_0 \in \mathcal{M}(\mathbb{R})$. Hence, the modified equation (91) has a global solution. Notice that if we let

$$m_0(x) = \sum_{i=1}^N \delta(x - c_i), \quad \text{and} \quad x_i^\epsilon(t) = X(c_i, t),$$

then System (95) for $\{x_i^\epsilon(t)\}_{i=1}^N$ recovers System (20).

Next, we use Equation (91) to justify that our regularization method has dispersive effects. For a smooth function f, we have

$$\rho_\epsilon * f(x) = \int_{\mathbb{R}} f(x - \epsilon y)\rho(y)\, dy = f(x) + a\epsilon^2 f_{xx}(x) + O(\epsilon^4),$$

where a is a constant given by

$$a = \frac{1}{2}\int_{\mathbb{R}} \rho(y)y^2 dy.$$

Using the above fact, we have

$$U_\epsilon = (\rho_\epsilon * u)^2 - (\rho_\epsilon * u_x)^2 = u^2 - u_x^2 + 2a\epsilon^2(uu_{xx} - u_x u_{xxx}) + O(\epsilon^4),$$

and

$$\begin{aligned} U^\epsilon &= U_\epsilon - a\epsilon^2 U_{\epsilon xx} + O(\epsilon^4) \\ &= u^2 - u_x^2 + a\epsilon^2[2(uu_{xx} - u_x u_{xxx}) + (u^2 - u_x^2)_{xx}] + O(\epsilon^4). \end{aligned}$$

Hence, the modified equation (91) becomes:

$$m_t + [m(u^2 - u_x^2)]_x + a\epsilon^2[2m(uu_{xx} - u_x u_{xxx}) + m(u^2 - u_x^2)_{xx}]_x + O(\epsilon^4) = 0. \tag{96}$$

To see that the correction term in the modified equation has dispersive effects, we do linearization around the constant solution 1. Let $u = 1 + \delta v$. We have

$$m = u - u_{xx} = 1 + \delta v - \delta v_{xx} = 1 + \delta n,$$

where $n = v - v_{xx}$. Keeping orders up to $O(\epsilon^2)$ and δ, we have the following linearized equation:

$$v_t + (2v + n)_x + 4a\epsilon^2 v_{xxx} + O(\delta) + O(\epsilon^4) = 0. \tag{97}$$

The leading term corresponding to the mollification is a dispersive term $4a\epsilon^2\delta v_{xxx}$. Hence, our regularization method has dispersive effects.

Appendix A. Proofs of Proposition 2.7 and 3.5.

Proof of Proposition 2.7. Because $\sum_{j=1}^N p_j G(x - x_j)$ is continuous, we have

$$\lim_{\epsilon\to 0} \rho_\epsilon * (u^{N,\epsilon})^2(x_i) = \left(\sum_{j=1}^N p_j G(x_i - x_j)\right)^2. \tag{98}$$

Next we estimate the second term $[\rho_\epsilon * (u_x^{N,\epsilon})^2](x_i)$ in $U^\epsilon(x_i)$. We have

(99)

$$\begin{aligned}(u_x^{N,\epsilon})^2(x) = \left(\sum_{j\in\mathcal{N}_{i1}} p_j G_x^\epsilon(x - x_j)\right)^2 + 2\sum_{j\in\mathcal{N}_{i1},k\in\mathcal{N}_{i2}} p_j G_x^\epsilon(x - x_j) p_k G_x^\epsilon(x - x_k)\\ + \left(\sum_{k\in\mathcal{N}_{i2}} p_k G_x^\epsilon(x - x_k)\right)^2 =: F_1^\epsilon(x) + F_2^\epsilon(x) + F_3^\epsilon(x).\end{aligned}$$

Because $G_x(x)$ is continuous at $x_i - x_j$, we have the following estimate for F_1^ϵ

$$\lim_{\epsilon\to 0} (\rho_\epsilon * F_1^\epsilon)(x_i) = \left(\sum_{j\in\mathcal{N}_{i1}} p_j G_x(x_i - x_j)\right)^2. \tag{100}$$

Because G and ρ_ϵ are even functions, we know G_x^ϵ is an odd function. Next, consider the second term F_2^ϵ on the right hand side of (99). Due to $x_k = x_i$ for $k \in \mathcal{N}_{i2}$, we have

$$
\begin{aligned}
(\rho_\epsilon * F_2^\epsilon)(x_i) =& 2 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_{\mathbb{R}} \rho_\epsilon(x_i - y) G_x^\epsilon(y - x_j) G_x^\epsilon(y - x_i) dy \\
=& 2 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_0^\infty \rho_\epsilon(y) G_x^\epsilon(-y) \\
& \times \left(\int_{\mathbb{R}} \Big[G_x(x_i - x_j - y - x) - G_x(x_i - x_j + y - x) \Big] \rho_\epsilon(x) dx \right) dy \\
\leq& 2 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_0^{\sqrt{\epsilon}} \rho_\epsilon(y) G_x^\epsilon(-y) \\
& \times \left(\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \Big| G_x(x_i - x_j - y - x) - G_x(x_i - x_j + y - x) \Big| \rho_\epsilon(x) dx \right) dy \\
& + 3 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_{\sqrt{\epsilon}}^\infty \rho_\epsilon(y) dy =: I_1^\epsilon + I_2^\epsilon.
\end{aligned} \tag{101}
$$

Due to $x_j \neq x_i$ for $j \in \mathcal{N}_{i1}$, we can choose ϵ small enough such that

$$
(x_i - x_j - y - x)(x_i - x_j + y - x) > 0, \text{ for } |x|, |y| < \sqrt{\epsilon}.
$$

Hence,

$$
|G_x(x_i - x_j - y - x) - G_x(x_i - x_j + y - x)| \leq \frac{1}{2}|2y| < \sqrt{\epsilon}.
$$

Putting the above estimate into I_1^ϵ gives

$$
\begin{aligned}
I_1^\epsilon =& 2 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_0^{\sqrt{\epsilon}} \rho_\epsilon(y) G_x^\epsilon(-y) \\
& \times \left(\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \Big| G_x(x_i - x_j - y - x) - G_x(x_i - x_j + y - x) \Big| \rho_\epsilon(x) dx \right) dy \\
\leq& \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} |p_j p_k| \cdot \sqrt{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned} \tag{102}
$$

For I_2^ϵ, changing variable gives

$$
\begin{aligned}
I_2^\epsilon =& 3 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_{\sqrt{\epsilon}}^\infty \rho_\epsilon(y) dy \\
=& 3 \sum_{j\in\mathcal{N}_{i1}, k\in\mathcal{N}_{i2}} p_j p_k \int_{\frac{1}{\sqrt{\epsilon}}}^\infty \rho(y) dy \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned} \tag{103}
$$

Combining (101), (102), and (103), we have

$$
\lim_{\epsilon\to 0} |(\rho_\epsilon * F_2^\epsilon)(x_i)| = 0. \tag{104}
$$

For F_3^ϵ in (99), using Lemma 2.5 we can obtain

$$\lim_{\epsilon\to 0}(\rho_\epsilon * F_3^\epsilon)(x_i) = \lim_{\epsilon\to 0}\int_{\mathbb{R}} \rho_\epsilon(x_i-y)\left(\sum_{k\in\mathcal{N}_{i2}} p_k \int_{\mathbb{R}} G(y-x_k-x)\rho_\epsilon(x)dx\right)^2 dy$$

$$= \left(\sum_{k\in\mathcal{N}_{i2}} p_k\right)^2 \lim_{\epsilon\to 0}\int_{\mathbb{R}} \rho_\epsilon(y)\Big(\int_{\mathbb{R}} G(y-x)\rho_\epsilon(x)dx\Big)^2 dy$$

$$= \left(\sum_{k\in\mathcal{N}_{i2}} p_k\right)^2 \lim_{\epsilon\to 0}\left[(G_x^\epsilon)^2 * \rho_\epsilon\right](0)$$

$$= \frac{1}{12}\left(\sum_{k\in\mathcal{N}_{i2}} p_k\right)^2, \tag{105}$$

where we used $x_i = x_k$ for $k\in\mathcal{N}_{i2}$ in the second step. Finally, combining (100), (104) and (105) gives

$$\lim_{\epsilon\to 0}[\rho_\epsilon * (u_x^{N,\epsilon})^2](x_i) = \frac{1}{12}\left(\sum_{k\in\mathcal{N}_{i2}} p_k\right)^2 + \left(\sum_{j\in\mathcal{N}_{i1}} p_j G_x(x_i-x_j)\right)^2. \tag{106}$$ ◻

Combining (98) and (106) gives (41).

Proof of Proposition 3.5. Let

$$4\int_{-\infty}^{\infty}\rho_\epsilon(x)\big[f_1^\epsilon(x)f_2^\epsilon(x) - f_1^\epsilon(s+x)f_2^\epsilon(s+x)\big]dx =: I_1^\epsilon - I_2^\epsilon,$$

where

$$I_1^\epsilon := 4\int_{-\infty}^{\infty}\rho_\epsilon(x)f_1^\epsilon(x)f_2^\epsilon(x)dx \quad\text{and}\quad I_2^\epsilon := 4\int_{-\infty}^{\infty}\rho_\epsilon(x)f_1^\epsilon(s+x)f_2^\epsilon(s+x)dx.$$

For I_1^ϵ, by changing of variables, we have

$$I_1^\epsilon = \int_{-\infty}^{\infty}\rho(x)\Big(\int_{-\infty}^{x}\rho(y)e^{\epsilon(y-x)}dy\Big)\Big(\int_{x}^{\infty}\rho(y)e^{\epsilon(x-y)}dy\Big)dx.$$

Set

$$F(x) := \int_{-\infty}^{x}\rho(y)dy.$$

By Lebesgue Dominated convergence Theorem, we have

$$\lim_{\epsilon\to 0} I_1^\epsilon = \int_{-\infty}^{\infty}\rho(x)\Big(\int_{-\infty}^{x}\rho(y)dy\Big)\Big(\int_{x}^{\infty}\rho(y)dy\Big)dx$$

$$= \int_{-\infty}^{\infty}F'(x)F(x)(1-F(x))dx = \frac{1}{6}. \tag{107}$$

Similarly, for I_2^ϵ we have

$$I_2^\epsilon = \int_{-\infty}^{\infty}\rho(x)\Big(\int_{-\infty}^{x+\frac{s}{\epsilon}}\rho(y)e^{\epsilon(y-x)-s}dy\Big)\Big(\int_{x+\frac{s}{\epsilon}}^{\infty}\rho(y)e^{\epsilon(x-y)+s}dy\Big)dx.$$

When $\delta > 0$ and $s \in [\delta, +\infty)$, we have $\dfrac{\delta}{\epsilon} \le \dfrac{s}{\epsilon}$. Hence,

$$0 < I_2^\epsilon \le \int_{-\infty}^{\infty} \rho(x)\Big(\int_{-\infty}^{\infty} \rho(y)dy\Big)\Big(\int_{x+\frac{s}{\epsilon}}^{\infty} \rho(y)dy\Big)dx$$
$$\le \int_{-\infty}^{\infty} \rho(x)\Big(\int_{x+\frac{\delta}{\epsilon}}^{\infty} \rho(y)dy\Big)dx.$$

Therefore, the following convergence holds uniformly for $s \in [\delta, +\infty)$:

$$\lim_{\epsilon\to 0} I_2^\epsilon = 0. \tag{108}$$

Combining (107) and (108) gives (71). □

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