ULTRA-CONTRACTIVITY FOR KELLER-SEGEL MODEL WITH DIFFUSION EXponent $m > 1 - 2/d$

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Abstract. This paper establishes the hyper-contractivity in $L^\infty(\mathbb{R}^d)$ (it’s known as ultra-contractivity) for the multi-dimensional Keller-Segel systems with the diffusion exponent $m > 1 - 2/d$. The results show that for the supercritical and critical case $1 - 2/d < m \leq 2 - 2/d$, if $\|U_0\|_{d(2-m)/2} < C_{d,m}$ where $C_{d,m}$ is a universal constant, then for any $t > 0$, $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)}$ is bounded and decays as $t$ goes to infinity. For the subcritical case $m > 2 - 2/d$, the solution $u(\cdot, t) \in L^\infty(\mathbb{R}^d)$ with any initial data $U_0 \in L^1_+(\mathbb{R}^d)$ for any positive time.

1. Introduction and main theorem. We consider the Keller-Segel model in spatial dimension $d \geq 3$:

\[
\begin{aligned}
    u_t &= \Delta u^m - \nabla \cdot (u \nabla c) , \quad x \in \mathbb{R}^d, \quad t \geq 0, \\
    -\Delta c &= u , \quad x \in \mathbb{R}^d , \quad t \geq 0 , \\
    u(x,0) &= U_0(x) \geq 0 , \quad x \in \mathbb{R}^d ,
\end{aligned}
\]

(1.1)

where the diffusion exponent $m$ is supercritical $0 < m < 2 - 2/d$, critical $m_c := 2 - 2/d$, and subcritical $m > 2 - 2/d$ respectively. This model was proposed by Keller

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\[ c(x, t) = c_d \int_{\mathbb{R}^d} \frac{u(y, t)}{|x - y|^{d-2}} dy, \quad (1.2) \]

where
\[ c_d = \frac{1}{d(d-2)\alpha_d}, \quad \alpha_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}, \quad (1.3) \]
\( \alpha_d \) is the volume of \( d \)-dimensional unit ball. The case \( m > 1 \) is called slow diffusion and the case \( m < 1 \) is called fast diffusion \([19, 20, 8]\).

The main characteristic of equation (1.1) is the competition between the diffusion and the nonlocal aggregation. This is well represented by the free energy for \( m > 1 \)
\[ F(u) = \frac{1}{m - 1} \int_{\mathbb{R}^d} u^m(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} u(x) dx. \quad (1.4) \]

For \( m = 1 \), the first term of (1.4) is replaced by \( \int_{\mathbb{R}^d} u \log u \ dx \) \([16]\). According to different \( m \), the competition results in different behaviors. Taking the mass invariant scaling \( u_\lambda(x) = u(x/\lambda, \lambda t) \) into account we can observe that for the supercritical case \( 1 \leq m < 2 - 2/d \), the aggregation dominates the diffusion for high density (large \( \lambda \)) and the density has finite-time blow-up \([11, 12, 6, 17, 16, 4]\). While for low density (small \( \lambda \)), the diffusion dominates the aggregation and the density has infinite-time spreading \([17, 18, 16, 2]\). On the contrary, for the subcritical case \( m > 2 - 2/d \), the aggregation dominates the diffusion for low density and prevents spreading, while for high density, the diffusion dominates the aggregation thus blow-up is precluded \([17, 18, 14]\).

In this paper, we mainly focus on the hyper-contractivity for the Keller-Segel model with \( m \leq 2 - 2/d \) and \( m > 2 - 2/d \) respectively. For non-degenerate Keller-Segel equation with \( m = 1, d = 2 \), Blanchet, Dolbeault and Perthame \([5]\) showed that if the initial data \( \|U_0\|_1 < 8\pi \) and \( U_0 \log U_0 \in L^1(\mathbb{R}^d) \), then for any \( 1 < q < \infty \) and any \( t > 0 \), there exists a continuous function \( h_q(t) \) satisfying that for \( t \to 0 \)
\[ h_q(t) \to \infty \]
and
\[ \|u(\cdot, t)\|_{L^q(\mathbb{R}^d)} \leq \|U_0\|_1 h_q(t). \]

Later, in 2012, Calvez, Corrias and Ebde \([7]\) proved the local in time hyper-contractive property for \( m = 1, d \geq 3 \), it reads that if \( U_0 \in (L^1 \cap L^a)(\mathbb{R}^d) \), \( a > d/2 \) arbitrarily close to \( d/2 \), there exists a finite time \( T_a = C(a) (\int_{\mathbb{R}^d} U_0^a dx)^{-\frac{1}{a-1}} \) and a local weak solution \( u \in L^\infty \left( (0, T_a); (L^1 \cap L^a)(\mathbb{R}^d) \right) \) satisfying that for any \( a < q < \infty \), there exists a constant \( C \) not depending on \( \|U_0\|_{L^q(\mathbb{R}^d)} \) such that
\[ \int_{\mathbb{R}^d} u(\cdot, t)^q dx \leq C \left(1 + t^{1-q}\right), \ \text{a.e. } t \in (0, T_a). \]

For general \( m \), in our previous paper \([3]\), it is showed that for \( 0 < m \leq 2 - 2/d \), if the initial data \( \|U_0\|_{L^2(\mathbb{R}^d)} < C_{d,m} \) where \( C_{d,m} \) is a universal constant depending on \( d, m \), then there exists a global weak solution. Furthermore, for \( 0 < m < 1 - 2/d \), the solution will vanish at finite time, and for \( m = 1 - 2/d \), the \( L^q(1 < q < \infty) \) norm has exponentially decay in time with the initial data in \( L^q \) norm. On the other hand, for supercritical and critical case \( 1 - 2/d < m \leq 2 - 2/d \), the solution satisfies
constant. For the subcritical case \( m > 2 - 2/d \), if the initial data \( U_0 \in L^1_+(\mathbb{R}^d) \), then the solution will be bounded in \( L^p(\mathbb{R}^d) \) for any \( 1 < p < \infty \).

For the hyper-contractive property in \( L^\infty \) norm (it’s also known as ultra-contractivity [10]), Corrias and Perthame [9] proved the hyper-contractivity for the parabolic-parabolic Keller-Segel model \((d \geq 3)\)
\[
\begin{cases}
    u_t = \Delta u - \nabla \cdot (u \nabla c), & x \in \mathbb{R}^d, \ t \geq 0, \\
    c_t - \Delta c = u - c, & x \in \mathbb{R}^d, \ t \geq 0, \\
    u(x, 0) = U_0(x) \geq 0, & x \in \mathbb{R}^d.
\end{cases}
\tag{1.5}
\]

The results show that if \( U_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d), \ d/2 < a \leq d, \ \nabla c_0 \in L^d(\mathbb{R}^d), \) there is a constant \( C(d, a) \) such that for
\[
\|U_0\|_{L^\infty(\mathbb{R}^d)} + \|\nabla c_0\|_{L^d(\mathbb{R}^d)} \leq C(d, a),
\]
the parabolic-parabolic system has a weak solution satisfying the hyper-contractivity type estimate for any \( \epsilon > 0 \)
\[
\|u(\cdot, t) - G(t) * U_0\|_{L^\infty(\mathbb{R}^d)} \leq C t^{\frac{1}{2} - d + \epsilon}, \ t \to \infty,
\]
where \( G(t) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} \) is the heat kernel. In this paper, we will extend the hyper-contractivity result in [3] to \( L^\infty \) norm for general \( m \). The main results are given below

**Theorem 1.1.** Let \( d \geq 3, \ p = \frac{d(2-m)}{2} \) and \( m > 1 - 2/d \). Assume \( U_0 \in L^1_+(\mathbb{R}^d), \)

(i) For the supercritical case and the critical case \( 1 - 2/d < m \leq 2 - 2/d, \) denote \( \eta := C_{d,m}^2 - \|U_0\|_{L^2}^2 \) where \( C_{d,m} \) is a universal constant given by (3.1), if \( \eta > 0 \), then there exists a global weak solution of (1.1) satisfying that for \( 0 < t \leq 1 \)
\[
\|u(\cdot, t)\|_{L^\infty}\leq \max\left[1, C(\eta, \|U_0\|_{L^1}, m, d)\left(\frac{1}{t^{\frac{4m(p+\epsilon_0)}{p(2m+p)+m-1, d)}} \cdot \frac{1}{m+d-1} + \frac{1}{m+d+1}\right)\cdot \frac{1}{\sqrt{t}}\right], \tag{1.6}
\]
and for \( 1 < t < \infty \)
\[
\|u(\cdot, t)\|_{L^\infty}\leq \max\left[1, C(\eta, \|U_0\|_{L^1}, m, d)\left(\frac{1}{t^{\frac{4m(p+\epsilon_0)}{p(2m+p)+m-1, d)}} \cdot \frac{1}{m+d-1} + \frac{1}{m+d+1}\right)\right], \tag{1.7}
\]
where \( \epsilon_0 \) satisfies \( \frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2, d} - \|U_0\|_{L^p}^2 = \frac{\eta}{2} \).

(ii) For the subcritical case \( m > 2 - 2/d, \) if \( m = 2, \) we also assume \( U_0 \log U_0 \in L^1(\mathbb{R}^d) \) and if \( m > 2, \) we assume \( U_0 \in L^m(\mathbb{R}^d) \), then
\[
\|u(\cdot, t)\|_{L^\infty}\leq \max\left[1, C(\|U_0\|_{L^1}, m, d)\left(1 + \frac{1}{m+d+1}\right)\cdot \frac{1}{\sqrt{t}}\right], \ 0 < t \leq 1,
\]
\[
\|u(\cdot, t)\|_{L^\infty}\leq \max\left[1, C(\|U_0\|_{L^1}, m, d)\left(1 + \frac{1}{m+d+1}\right)\right], \ 1 < t < \infty.
\]
Furthermore, for any \( T > t_0 > 0, \) the weak solution has the following regularities
\[
u(x, t) \in L^\infty(t_0, T; L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \cap L^2(t_0, T; H^1(\mathbb{R}^d)), \tag{1.8}
\]
and
\[ u_t \in L^p (0, T; W_{loc}^{-1,p_1} (\mathbb{R}^d)) \cap L^2 (t_0, T; H^{-1} (\mathbb{R}^d)) \] for some \( p_1, p_2 \geq 1 \). \hfill (1.9)

This paper is organized as follows. In Section 2, we list some preliminary lemmas which will be used to prove the \( L^\infty \) norm. Section 3 is devoted to show the main theorem on hyper-contractive property in \( L^\infty (\mathbb{R}^d) \). Finally, Section 4 considers the boundedness in \( L^\infty (\mathbb{R}^d) \) uniformly in time.

2. Preliminary. Before proving hyper-contractive estimates, we need the following preparations, some lemmas have been proved in [3].

Lemma 2.1. Let \( 1 < \frac{b}{a} < \frac{2d}{a(d-2)} \) and \( \frac{b}{a} < \frac{2}{a} + \frac{2}{d} \). Assume \( w \in L^1_+ (\mathbb{R}^d) \) and \( w^{1/a} \in H^1 (\mathbb{R}^d) \) with \( a > 0 \), then
\[
\|w\|_{b/a}^{b/a} \leq C(\delta)C_0^{-\gamma} \|w\|_1^{\gamma} + C_0 \|\nabla w^{1/a}\|_2^2,
\]
where
\[
\delta = 2 \left( \frac{1}{a} - \frac{d-2}{2d} \right), \quad \gamma = 1 + \frac{2b-2a}{2d-bd+2a},
\]
and \( C(\delta) = \delta^{-\frac{1}{\delta-1}} S_d^{-\frac{b\theta d}{2}} \) with \( \theta = \frac{1}{a} - \frac{d-2}{2d} \) and \( \delta' = \frac{\delta}{\delta-1} \). \( C_0 \) is an arbitrary positive constant.

Proof. The Sobolev inequality reads as follows
\[
S_d \|u\|_{2d/(d+2)}^2 \leq \|\nabla u\|_2^2, \quad S_d = \frac{d(d-2)}{4} 2^{2/d+1+1/d} \left( \frac{d+1}{2} \right)^{-2/d}, \hfill (2.1)
\]

Taking \( u = w^{1/a} \) in (2.1) and the interpolation inequality with \( 1 < \frac{b}{a} < \frac{2d}{a(d-2)} \) yields
\[
\|w\|_{b/a} \leq \|w\|_1^{1-\theta} \|w\|_2^{\theta} S_d^{-\frac{b\theta d}{2}} \|w\|_1^{1-\theta} \|w^{1/a}\|_2^\theta \leq S_d^{-\theta d/2} \|w\|_1^{1-\theta} \|\nabla w^{1/a}\|_2^\theta,
\]
whence follows
\[
\|w\|_{b/a}^{b/a} \leq C(d) \|w\|_1^{(1-\theta)b/a} \|\nabla w^{1/a}\|_2^b, \hfill (2.2)
\]

where
\[
\theta = \frac{1}{a} - \frac{b}{d} + \frac{2}{d}, \quad C(d) = S_d^{-b\theta/2}.
\]

It is easy to verify that \( b\theta < 2 \) if \( \frac{b}{a} < \frac{2}{a} + \frac{2}{d} \). Therefore, by the Young inequality we have
\[
\|w\|_{b/a}^{b/a} \leq C(d)^{\beta - \delta'} \frac{b}{\delta'} \|w\|_1^{(1-\theta)\delta'} + \frac{\beta \delta}{\delta'} \|\nabla w^{1/a}\|_2^{\beta \delta},
\]
here \( \delta' = \frac{\delta}{\delta-1} \) and \( b\theta\delta = 2 \) such that
\[
\delta = 2 \left( \frac{1}{a} - \frac{d-2}{2d} \right) \frac{b}{b/a - 1}.
\]

Let \( C_0 = \frac{\beta \delta}{\delta'} \delta' \) and thus \( \beta - \delta' = (C_0 \delta')^{-\frac{1}{2}} \frac{a}{\delta-1} \). We denote \( C(\delta) = \delta^{-\frac{1}{\delta-1}} C(d)^{\beta \delta'}, \gamma = \frac{b}{a} (1-\theta)\delta' \), this concludes the proof. \( \square \)
Now taking
\[ a = \frac{2}{m + q - 1}, \quad b = \frac{2q}{m + q - 1}; \quad C_0 = \frac{2mq(q - 1)}{(m + q - 1)^2}, \quad w = u \]
in Lemma 2.1 we obtain the following lemma

**Lemma 2.2.** Let \( d \geq 3, q > 1, m > 1 - 2/d \), assume \( u \in L_+^1(\mathbb{R}^d) \) and \( u^{\frac{m+q-1}{2}} \in H^1(\mathbb{R}^d) \), then
\[
\left( \|u\|_q \right)^{1 + \frac{m+q-2/d}{q-1}} \leq S_d^{-1} \|\nabla u^{(q + m - 1)/2}\|_2 \|u\|^\frac{1}{q} \left( q + m - 1 \right). \tag{2.3}
\]
and
\[
\|u\|_q^2 \leq \frac{2mq(q - 1)}{(m + q - 1)^2} \|\nabla u^{\frac{m+q-1}{2}}\|_2^2 + \left( 1 - \frac{\alpha_0}{2} \right) [S_d \frac{2mq(q - 1)}{(m + q - 1)^2} \alpha]^{\frac{1}{1 - \eta}} \|u\|_{\eta},
\]
where \( \eta = 1 + \frac{2(q - 1)}{dm + 2q - 2d} \), \( \alpha_0 = \frac{2(q - 1)}{m + q - 2 + 2d} < 2 \) for \( m > 1 - 2/d \).

Similarly letting
\[ a = \frac{2}{m + q - 1}, \quad b = \frac{2(q + 1)}{m + q - 1}, \quad C_0 = \frac{2mq}{(m + q - 1)^2} \]
in Lemma 2.1 leads to

**Lemma 2.3.** Let \( d \geq 3, q > 0, m > 2 - 2/d \), assume \( u \in L_+^1(\mathbb{R}^d) \) and \( u^{\frac{m+q-1}{2}} \in H^1(\mathbb{R}^d) \), then
\[
\|u\|_{q+1}^2 \leq \frac{2mq}{(m + q - 1)^2} \|\nabla u^{\frac{m+q-1}{2}}\|_2^2 + \left( 1 - \frac{\alpha}{2} \right) [S_d \frac{2mq(q - 1)}{(m + q - 1)^2} \alpha]^{\frac{1}{1 - \eta}} \|u\|_{\eta},
\]
where \( \eta = 1 + \frac{2q}{dm + 2q + 2d} \), \( \alpha = \frac{2q}{m + q - 2 + 2d} < 2 \) for \( m > 2 - 2/d \).

For the supercritical case \( 0 < m < 2 - 2/d \), choosing particular \( a, b \) in (2.2) of Lemma 2.1 and using the Young inequality will be used in the next sections.

**Lemma 2.4.** Let \( d \geq 3, 0 < m \leq 2 - 2/d \), \( p = \frac{d(2-m)}{q} \), \( q \geq p \) and \( u \in L_+^1(\mathbb{R}^d) \).

Then
\[
\|u\|_{q+1}^2 \leq S_d^{-\frac{2}{q}} \|\nabla u^{(m+q-1)/2}\|_2 \|u\|_{\frac{2-m}{p}}, \tag{2.4}
\]
and for \( q \geq r > p \)
\[
\|u\|_{q+1}^2 \leq S_d^{-\frac{2}{q}} \left\| \nabla u^{\frac{m+q-1}{2}} \right\|_2^\alpha \|u\|_{r}^\beta \leq \frac{2mq}{(m + q - 1)^2} \left\| \nabla u^{\frac{m+q-1}{2}} \right\|_2^2 + C(q, r, d) \left( \|u\|_{r} \right)^{\delta}, \tag{2.5}
\]
where
\[
\alpha = \frac{2(q - r + 1)}{q - r + 1 + 2(r - p)/d} < 2, \quad \beta = q + 1 - \frac{m + q - 1}{2} \alpha,
\]
\[
\delta = \frac{\beta}{r(1 - \alpha/2)} = 1 + \frac{1 + q - r}{r - p},
\]
\[
C(q, r, d) = \left[ \frac{2mq(q - r + 1 + 2(r - p)/d)}{S_d^{\frac{2}{q}}(q + m - 1)^2(q - r + 1)} \right] \frac{d(q-r+1)}{d(q-r+1)+2(r-p)}.
\]

Now we define the weak solution which we will deal with throughout this paper.
Definition 2.5. (Weak solution) Let $U_0 \in L^1_+(\mathbb{R}^d)$ be the initial data and $T \in (0, \infty)$. $c$ is the concentration associated with $u$. $u$ is a weak solution to the system (1.1) with initial data $U_0$ and it satisfies:

(i) Regularity:

$$u \in L^{\max(m, 2)} \left(0, T; L^1_+ \cap L^{\max(m, \frac{2d}{d-2})}(\mathbb{R}^d)\right),$$

$$\partial_t u \in L^{p_2} \left(0, T; W^{-1, p_1}_{loc}(\mathbb{R}^d)\right) \text{ for some } p_1, p_2 \geq 1.$$  

(ii) For $\forall \psi \in C_0^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$

$$\int_{\mathbb{R}^d} \psi(x) dx - \int_{\mathbb{R}^d} \psi U_0 dx = \int_0^t \int_{\mathbb{R}^d} \Delta \psi u dx ds$$

$$- \frac{c_d(d-2)}{2} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\nabla \psi(x) \cdot (x-y)}{|x-y|^2} u(x,s) u(y,s) dx dy ds.$$  

Remark 1. Notice that the regularity (2.6) is enough to make sense of each term in (2.8). By the HLS inequality [15] one has

$$\left\| u(x) \right\|_{2d/(d+2)}^2 < \infty.$$

Before showing the global existence results for $0 < m < 2 - 2/d$, we need the following lemma.

Lemma 2.6. Assume $y(t) \geq 0$ is a $C^1$ function for $t > 0$ satisfying $y'(t) \leq \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then

(i) For $a > 1$, $y(t)$ has the following hyper-contractive property

$$y(t) \leq (\alpha/\beta)^{1/a} \left[ \frac{1}{\beta(a-1)t} \right]^{\frac{1}{a-1}}, \quad \text{for } t > 0.$$  

Furthermore, if $y(0)$ is bounded, then

$$y(t) \leq \max \left( y(0), (\alpha/\beta)^{1/a} \right).$$

(ii) For $a = 1$, $y(t)$ decays exponentially

$$y(t) \leq \frac{\alpha}{\beta} + y(0)e^{-\beta t}.$$  

(iii) For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, that’s there exists a $0 < T_{\text{ext}} \leq \frac{y^{1-a}(0)}{\beta(1-a)}$ such that $y(t) = 0$ for all $t > T_{\text{ext}}$.

Proof. The lemma was proved in [3] except (2.10), here we give the proof for (2.10). The ODE inequality can be recast as

$$y'(t) \leq \beta \left[ (\alpha/\beta)^{\frac{1}{a}} - y(t)^a \right].$$

Case 1. If $y(0) \leq (\alpha/\beta)^{1/a}$, then by contradiction arguments we have that for any $t > 0$

$$y(t) \leq (\alpha/\beta)^{1/a}.$$
Case 2. For $y(0) > (\alpha/\beta)^{1/a}$, if $y(t) > (\alpha/\beta)^{1/a}$ for all $t > 0$, then $y'(t) < 0$ and thus $y(t) < y(0)$. Otherwise, denote $t_0$ as the first time such that $y(t_0) = (\alpha/\beta)^{1/a}$, then

$$y'(t) < 0, \quad 0 \leq t < t_0,$$

$$y(t) \leq (\alpha/\beta)^{1/a}, \quad t > t_0.$$ 

Collecting the two cases we obtain

$$y(t) \leq \max \left( y(0), (\alpha/\beta)^{1/a} \right).$$

3. The hyper-contractive estimates and proof of the main theorem. In this section, we will show the hyper-contractive property for $m > 1 - 2/d$. Firstly we define a constant which is related to the initial condition for the existence results:

$$C_{d,m} := \left( \frac{4mp}{(m + p - 1)^2 S_d^1} \right)^{\frac{1}{m}}, \quad p = \frac{d(2 - m)}{2},$$ (3.1)

where $S_d$ is given by (2.1). The following theorems give the hyper-contractive of $L^q$ for any $1 < q < \infty$ which is proved in [3]. For the supercritical and critical cases,

**Theorem 3.1** ([3]). Let $d \geq 3$, $0 < m \leq 2 - 2/d$ and $p = \frac{d(2 - m)}{2}$, $\eta := (2 - m) - \|U_0\|_p^2 - m$. Assume $U_0 \in L^1_{\text{ext}}(\mathbb{R}^d)$ and $\eta > 0$, then there exists a global weak solution $u$ such that $\|u(\cdot, t)\|_p < C_{d,m}$ for all $t \geq 0$. Furthermore,

(i) For $0 < m < 1 - 2/d$, there exists a minimal extinction time $T_{\text{ext}}(\|U_0\|_1, \eta, p)$ such that the weak solution vanishes a.e. in $\mathbb{R}^d$ for all $t \geq T_{\text{ext}}$.

(ii) For $m = 1 - 2/d$, the weak solution decays exponentially

$$\|u(\cdot, t)\|_p \leq \|U_0\|_p e^{-\frac{\eta}{\|U_0\|_1^{(p-1)/p}} t},$$ (3.2)

(iii) For $1 - 2/d < m \leq 2 - 2/d$, the weak solution satisfies mass conservation and the following hyper-contractive estimates hold true that for any $t > 0$ and $1 \leq q \leq p$

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^d)} \leq C(\eta, \|U_0\|_1, q) t^{-\frac{q-1}{m-1} + \epsilon_0},$$ (3.3)

and for $p < q < \infty$

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^d)} \leq C(\eta, \|U_0\|_1, q) \left( t^{-\frac{p+\epsilon_0-1}{q} + \epsilon_0} + t^{-\frac{q-1}{m-1} + \epsilon_0} \right),$$ (3.4)

where $\epsilon_0$ satisfies $\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^1} - \|U_0\|_p^{2-m} = \frac{\eta}{2}$.

**Theorem 3.2** ([3]). For $m > 2 - 2/d$, assume $U_0 \in L^1_{\text{ext}}(\mathbb{R}^d)$. Assume also $U_0 \log U_0 \in L^1(\mathbb{R}^d)$ for $m = 2$ and $U_0 \in L^{m-1}(\mathbb{R}^d)$ for $m > 2$, then there exists a weak solution globally in time satisfying the following hyper-contractive property that for any $1 < q < \infty$

$$\|u\|_q^q \leq C(\|U_0\|_1, q, m, d) + \left[ \frac{q-1}{t} \right]^{q-1}, \quad \text{for any } t > 0.$$ (3.5)
Using the boundedness of $\|u\|_q$ for any $1 < q < \infty$ we can prove our main result about the hyper-contractivity in $L^\infty$ estimates.

**Proof of Theorem 1.1.** The global existence of a weak solution was proved in [3]. Now we will give the proof of the hyper-contractivity in $L^\infty(\mathbb{R}^d)$ for any positive time. Firstly we denote $q_k := 3^k + m + d + 1$ and estimate $\int_{\mathbb{R}^d} u^{q_k} \, dx$.

**Step 1.** (The $L^{q_k}$ estimate) Multiplying equation (1.1) with $u^{q_k-1}(q_k > 1)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} \, dx = - \frac{4mq_k(q_k - 1)}{(q_k + m - 1)^2} \int_{\mathbb{R}^d} \left| \nabla u^{q_k + m - 1} \right|^2 \, dx + (q_k - 1) \int_{\mathbb{R}^d} u^{q_k + 1} \, dx,$n

(3.6)

from Lemma 2.1 by letting

$$a = \frac{2q_k - 1}{q_k + m - 1}, \quad b = \frac{2(q_k + 1)}{q_k + m - 1}, \quad w = u^{\frac{q_k + m - 1}{2}}$$

we obtain

$$\int_{\mathbb{R}^d} u^{q_k + 1} \, dx \leq C(\delta_1)C_1^{-\frac{1}{q_k - 1}} \left( \int_{\mathbb{R}^d} u^{q_k - 1} \, dx \right)^{\gamma_1} + C_1 \left\| \nabla u^{q_k + m - 1} \right\|_2,$n

(3.7)

where $\delta_1 = \frac{2(1 - \frac{d-2}{2})}{2} = O(1)$ and $\gamma_1 = 1 + \frac{2b - 2a}{2d - k d} \leq 3$ with $m > 0$, $C_1$ is a positive constant to be determined. It’s easy to verify that $1 < b/a < \frac{2d}{a(d-2)}$ and $b/a < 2/a + 2/d$.

Substituting (3.7) into (3.6) we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} \, dx \leq \left( C_1(q_k - 1) - \frac{4mq_k(q_k - 1)}{(q_k + m - 1)^2} \right) \int_{\mathbb{R}^d} \left| \nabla u^{q_k + m - 1} \right|^2 \, dx$$

$$+ C(\delta_1)(q_k - 1)C_1^{-\frac{1}{q_k - 1}} \left( \int_{\mathbb{R}^d} u^{q_k - 1} \, dx \right)^{\gamma_1}.$n

(3.8)

We can see that for $k \to \infty$,

$$\frac{4mq_k(q_k - 1)}{(q_k + m - 1)^2} \to 4m,$n

therefore, in order to control the term $\int_{\mathbb{R}^d} \left| \nabla u^{q_k + m - 1} \right|^2 \, dx$ in (3.8), since $q_k > m + 1$, we can choose $C_1 = \frac{m}{2(q_k - 1)}$, $C_2 = m/2$ such that

$$C_1(q_k - 1) - \frac{4mq_k(q_k - 1)}{(q_k + m - 1)^2} \leq -C_2,$n

(3.9)

this follows

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} \, dx \leq -C_2 \int_{\mathbb{R}^d} \left| \nabla u^{q_k + m - 1} \right|^2 \, dx$$

$$+ C(\delta_1)(q_k - 1)C_1^{-\frac{1}{q_k - 1}} \left( \int_{\mathbb{R}^d} u^{q_k - 1} \, dx \right)^{\gamma_1}.$n

(3.10)

On the other hand, from (2.2) of Lemma 2.1 letting

$$a = \frac{2q_k - 1}{q_k + m - 1}, \quad b = \frac{2q_k}{q_k + m - 1}$$

one has

$$(\|u\|_{q_k}^{1 + m_1 - 1/2}) \leq S_d^{-\frac{m_1 - 1}{q_k - m - 1}} \left( \int_{\mathbb{R}^d} u^{q_k - 1} \, dx \right)^{-\frac{1}{q_k - m - 1}}(2d/m + 1),$$n

where
substituting it into (3.10) follows that for any $t > t_0$ with fixed $t_0 > 0$
\[
\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} dx \leq - \frac{C_2}{S_d^{a-1} (\int_{\mathbb{R}^d} u^{q_k-1} dx)^{\frac{1}{q_k-1}}} \left( \|u\|_{\varphi_k}^{m-1+2q_k-1/d} \right)^{\frac{m-1+2q_k-1/d}{q_k-1}} + C(\delta_1) q_k^{1-\gamma_1} \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\gamma_1},
\]
where for $m > 0$
\[
\gamma_1 = 1 + \frac{2q_k - 2q_k - 1 + 2}{dm - 2d + 2q_k - 1} < 3, \quad \delta_1 = 1 + \frac{m - 2 + 2q_k - 1/d}{q_k - q_k - 1} \geq 1 + 1/d.
\]
Since $q_k > 1$, thus for any $t > t_0 > 0$ we have
\[
\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} dx \leq - \frac{C(m, d)}{\sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\frac{1}{q_k-1}}} \left( \|u\|_{\varphi_k}^{m-1+2q_k-1/d} \right)^{\frac{m-1+2q_k-1/d}{q_k-q_k-1}} + C(\delta_1) q_k^{d+1} \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\gamma_1},
\]
\textbf{Step 2. (Iterative procedures and hyper-contractive estimates)} By applying Lemma 2.6, letting $y_k(t) = \int_{\mathbb{R}^d} u^{q_k} dx$ and taking
\[
a = 1 + \frac{m - 1 + 2q_k - 1/d}{q_k - q_k - 1} \geq 1 + 1/d, \quad \text{if } m > 0,
\]
\[
\begin{align*}
\beta(t_0) &= \frac{C(m, d)}{\sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\frac{1}{q_k-1}}} \left( \|u\|_{\varphi_k}^{m-1+2q_k-1/d} \right)^{\frac{m-1+2q_k-1/d}{q_k-q_k-1}}, \\
\alpha(t_0) &= C(\delta_1) q_k^{d+1} \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\gamma_1},
\end{align*}
\]
in the ODE inequality (2.9), then
\[
y_k(t) \leq \left[ \alpha(t_0) / \beta(t_0) \right]^{1/a} + \left[ \frac{1}{\beta(t_0)(a-1)(t-t_0)} \right]^{1/(a-1)}, \quad t > t_0, \quad (3.11)
\]
plugging $a, \alpha(t_0), \beta(t_0)$ into (3.11) yields that for any $t > t_0 > 0$
\[
y_k(t) \leq C(m, d) q_k^{d+1} \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{\gamma_1 + \frac{2q_k-1/d-1}{q_k-1}} + \frac{C(m, d) \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)}{(a-1)(t-t_0)} \frac{m-1+2q_k-1}{q_k-q_k-1} \left[ \frac{m-1+2q_k-1}{q_k-q_k-1} \right]^{\gamma_1}
\]
\[
\leq C(m, d) q_k^{d+1} \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^{A} + \left[ \frac{C(m, d) \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)}{(t-t_0)^{1/\gamma_1}} \right]^B
\]
\[
\leq \max[1, C(m, d)][2(m + d + 1)3^k]^{d+1}.
\]
\[
\left( \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right)^A + \left[ \sup_{t_0 < t < \infty} \left( \int_{\mathbb{R}^d} u^{q_k-1} dx \right) / (t-t_0)^{1/\gamma_1} \right]^B \right)^{d+1}.
\]
where we have used \( a - 1 \geq 1/d \) and for \( m > 0 \)
\[
\eta_0 = \frac{2q_k/d + m - 1}{q_k - q_{k-1}} \geq \frac{d}{3},
\]
\[
A = \frac{\gamma_1 + \eta_0}{a} = 1 + \frac{2q_k - 2q_{k-1} + 2}{dm - 2d + 2} \frac{q_k - q_{k-1}}{q_k - q_{k-1}} \leq 3,
\]
\[
B = \frac{\eta_0}{a - 1} = \frac{2q_k/d + m - 1}{2q_k - 1/d + m - 1} \leq 3,
\]
declare \( C_0 = \max[1, C(m, d)]|2(m + d + 1)|^{d+1} \), from (3.12) one has that for any \( t_0 < t < \infty \)
\[
y_k(t) \leq C_0 3^{(d+1)k} \left[ \sup_{t_0 < t < \infty} y^A_{k-1}(t) + \left( \sup_{t_0 < t < \infty} y^{-1}(t) \right) \right]^{B}
\]
\[
\leq 2C_0 3^{(d+1)k} \max \left\{ 1, \sup_{t_0 < t < \infty} y^3_{k-1}(t), \left( \sup_{t_0 < t < \infty} y^{-1}(t) \right) \right\}^{3}. \quad (3.13)
\]

Next we will analyze the inequality (3.13).

If \( 0 < t \leq 1 \), take \( 0 < (t - t_0)^{1/\eta_0} < 1 \), then \( 1/\eta_0 \leq \frac{d}{3} \) gives rise to
\[
y_k(t) \leq 2C_0 3^{(d+1)k} \max \left\{ 1, \left( \sup_{t_0 < t < \infty} y^{-1}(t) \right) \right\}^{3}
\]
\[
\leq \frac{2C_0}{(t - t_0)^d} 3^{(d+1)k} \max \left\{ 1, \sup_{t_0 < t < \infty} y^3_{k-1}(t) \right\},
\]
then after some iterative procedures for any fixed \( t, t_0 \), we have
\[
y_k(t) \leq \left( \frac{2C_0}{(t - t_0)^d} \right)^{k-1} 3^{d+1}(\frac{k+1}{3} - \frac{1}{2} - \frac{1}{2}) \max \left( \sup_{t_0 < t < \infty} y^3_{k0}(t), 1 \right). \quad (3.14)
\]
Recalling \( q_k = 3^k + m + d + 1 \), taking the power \( 1/\eta_k \) to both sides of (3.14) we conclude that for \( t_0 < t \leq 1 \)
\[
\|u(\cdot, t)\|_{\infty} \leq \frac{\sqrt{2C_0}}{(t - t_0)^{d/2}} 3^{3(d+1)/4} \max \left( \sup_{t_0 < t < \infty} \|u(t)\|^{m+d+2}_{m+d+2}, 1 \right), \quad (3.15)
\]
take \( t_0 = t/2 \) we have
\[
\|u(\cdot, t)\|_{\infty} \leq \frac{C(d, m)}{t^{d/2}} 3^{3(d+1)/4} \max \left( \sup_{t/2 < s < \infty} \|u(s)\|^{m+d+2}_{m+d+2}, 1 \right), \quad 0 < t \leq 1. \quad (3.16)
\]
Similarly, if \( 1 < t < \infty \), taking \( t - t_0 > 1/2 \) in (3.13) we have
\[
y_k(t) \leq C_1 3^{(d+1)k} \max \left\{ 1, \sup_{t_0 < t < \infty} y^3_{k-1}(t) \right\},
\]
where \( C_1 = 2C_0 2^{1/\eta_0} \), this follows
\[
y_k(t) \leq C_1 3^{(d+1)k} \max \left( \sup_{t_0 < t < \infty} y^3_{k0}(t), 1 \right), \quad (3.17)
\]
taking the power $\frac{1}{q_k}$ to both sides of (3.17) we conclude that for $1/2 < t - t_0 < \infty$
\[
\|u(\cdot,t)\|_\infty \leq C_1 t_0^{d(1/4)} \max \left( \sup_{t_0 < t < \infty} \|u(t)\|_{m+d+2}^{m+d+2}, 1 \right),
\]
(3.18)

taking $t_0 = t/2$ in (3.18) follows
\[
\|u(\cdot,t)\|_\infty \leq C_1 t^{d(1/4)} \max \left( \sup_{t_0 < t < \infty} \|u(t)\|_{m+d+2}^{m+d+2}, 1 \right) < q \leq 1, \quad 1 < t < \infty,
\]
(3.19)

**Step 3.** (Boundedness and decay in $L^\infty$ norm for supercritical, critical cases) For $1 - 2/d < m \leq 2 - 2/d$, by virtue of (iii) of Theorem 3.1, due to $m + d + 2 > \frac{d(2-m)}{2} = p$ we have that for any $0 < t < \infty$
\[
\|u(t)\|_{m+d+2}^{m+d+2} \leq C(\eta, \|U_0\|_1, m, d) \cdot \left( \frac{1}{t} \frac{p+\eta_0-1}{\eta_0(p+\eta_0+1)m} \frac{m+d+1}{m+1/d} \right.
\]
\[
\quad \left. + \frac{1}{t} \frac{m+d+2-1}{(m+d+2)(m-1/d)} \right).
\]
\[
=C(\eta, \|U_0\|_1, m, d) \left( \frac{1}{t} \frac{p+\eta_0-1}{\eta_0(p+\eta_0+1)m} \frac{m+d+1}{m+1/d} \right.
\]
\[
\quad \left. + \frac{1}{t} \frac{m+d+1}{m-1/d} \right),
\]

where $\eta = C_{d,m} - \|U_0\|_p$ and $\epsilon_0$ satisfies $\frac{4m(p+\epsilon_0)}{p(p+\epsilon_0+1)S_d} = \eta/2$, then (3.16) and (3.19) follow the boundedness of the solution that for $0 < t < 1$
\[
\|u(\cdot,t)\|_\infty \leq \max \left[ 1, C(\eta, \|U_0\|_1, m, d) \right] \cdot \left( \frac{1}{t} \frac{p+\eta_0-1}{\eta_0(p+\eta_0+1)m} \frac{m+d+1}{m+1/d} \right.
\]
\[
\quad \left. + \frac{1}{t} \frac{m+d+1}{m-1/d} \right) t^{d/2},
\]

and for $1 < t < \infty$
\[
\|u(\cdot,t)\|_\infty \leq \max \left[ 1, C(\eta, \|U_0\|_1, m, d) \right] \left( \frac{1}{t} \frac{p+\eta_0-1}{\eta_0(p+\eta_0+1)m} \frac{m+d+1}{m+1/d} \right.
\]
\[
\quad \left. + \frac{1}{t} \frac{m+d+1}{m-1/d} \right).
\]

**Step 4.** (Boundedness in $L^\infty$ norm for subcritical case) For $m > 2 - 2d$, by Theorem 3.2, we have for any $1 < q < \infty$,
\[
\|u\|_q^q \leq C(\|U_0\|_1, q, m, d) \left[ \frac{q-1}{t} \right]^{q-1}, \quad \text{for any } t > 0.
\]
(3.20)

Similar to (3.16) and (3.18) we obtain
\[
\|u(\cdot,t)\|_\infty \leq \max \left[ 1, C(\|U_0\|_1, m, d) \right] \left( 1 + \frac{1}{t^{m+d+2-1}} \right) \cdot \frac{1}{t^{d/2}}, \quad 0 < t \leq 1,
\]
\[
\|u(\cdot,t)\|_\infty \leq \max \left[ 1, C(\|U_0\|_1, m, d) \right] \left( 1 + \frac{1}{t^{m+d+2-1}} \right), \quad 1 < t < \infty.
\]

**Step 5.** (Time regularity) Previously we have the following basic estimates that for any $T > 0$
\[
\|u\|_{L^\infty(0,T;L^2(R^d))} \leq C,
\]
(3.21)
\[
\left\| \nabla u \right\|_{L^2(0,T;L^p(R^d))} \leq C, \quad 1 < r \leq p,
\]
(3.22)
\[
\|u\|_{L^{p+1}(0,T;L^{p+1}(R^d))} \leq C.
\]
(3.23)
After some computations we obtain the time regularity
\[ \|u_t\|_{L^{\min\left(2,\frac{2(p+1)}{m+1}\right)}}_{0,T;W^{-1}_{loc}(\mathbb{R}^d)} \leq C. \] (3.24)

On the other hand, using the \(L^\infty\) bound for any \(t > t_0 > 0\), it’s easy to verify that for any \(T > t_0 > 0\)
\[ \|u\|_{L^\infty(t_0,T;L^1 \cap L^\infty(\mathbb{R}^d))} \leq C, \]
\[ \|\nabla u\|_{L^2(t_0,T;L^2(\mathbb{R}^d))} \leq C, \quad \text{for any } 1 < q < \infty, \]
here we can choose \(\frac{m+q-1}{2} \geq 1\) such that the solution satisfies the following gradient estimates
\[ \|\nabla u\|_{L^2(t_0,T;L^2(\mathbb{R}^d))} \leq C, \]
then some computations by using the above regularities verify the regularities (1.8) and (1.9). This ends the proof. \(\square\)

Using Theorem 2.17 in [3] for the local existence results directly leads to the following corollary.

**Corollary 1.** For \(0 < m < 2 - 2/d\), if \(U_0 \in L^1 \cap L^2(\mathbb{R}^d)\) for some \(m \leq q < \infty\) and \(p < q < \infty\), then there exists a finite time \(T_q\) depending on \(\|U_0\|_q\) and a local weak solution \(u(x,t)\) such that
\[ \|u(\cdot,t)\|_{L^\infty} \leq \frac{C(\|U_0\|_q, q)}{t^\alpha}, \quad 0 < t < T_q/2, \]
where \(\alpha\) is a positive constant.

Notice that the local existence results in Theorem 2.17 of [3] also hold true for \(m > 0\).

### 4. The uniform estimates in \(L^\infty(\mathbb{R}^d)\)

In this section, we will show that if \(U_0 \in L^1 \cap L^\infty(\mathbb{R}^d)\), then the solution is bounded in \(L^\infty(\mathbb{R}^d)\) uniformly in time instead of hypercontractivity in Section 3. Firstly, we will give the proof for the boundedness in \(L^q(\mathbb{R}^d)(1 < q < \infty)\) uniformly in time in the following proposition.

**Proposition 1.** Let \(d \geq 3\),

1. For \(0 < m \leq 2 - 2/d\) and \(p = \frac{d(2-m)}{2} \geq 1\), \(\eta := C^2 - m - \|U_0\|_p^{-m} = \eta\), if \(U_0 \in L^1 \cap L^\infty(\mathbb{R}^d)\) for some \(1 < q < \infty\) and \(\eta > 0\), then there exists a global weak solution \(u\) such that
\[ \|u\|^q_q \leq C(\|U_0\|_1, q)(\|U_0\|^{\frac{p-1}{p}}_p), \quad 1 < q \leq p, \]
\[ \|u\|^q_q \leq \|U_0\|^q_q + C(\|U_0\|_1, q)(\|U_0\|^q_q)^{\frac{p+1-\epsilon_0}{\epsilon_0}}^{\frac{2-p+1}{2-p+1}}, \quad p < q < \infty, \] (4.1)

where \(\epsilon_0\) satisfies
\[ \frac{4m(p+m)}{p+\epsilon_0 + m - 1}S^2_d - \|U_0\|_p^{-m} = \frac{\eta}{2}. \] (4.2)
2. For $m > 2 - 2/d$, if $U_0 \in L^1 \cap L^q(\mathbb{R}^d)$ for some $1 < q < \infty$, then

$$
\|u\|_q^2 \leq \|U_0\|_q^2 + \left(1 - \frac{\alpha_0}{2}\right) \left[ S_d \frac{2mq(q-1)}{(m+q-1)^2} \frac{1}{\alpha_0} \|U_0\|_1^{1 + \frac{2(q-1)}{m+q-2+2/d}} \right]^{\frac{1}{1-\alpha_0}} \|U_0\|_1^{1 + \frac{2(q-1)}{m+q-2+2/d}}
$$

$$
+ (q-1) \left(1 - \frac{\alpha_0}{2}\right) \left[ S_d \frac{2mq}{(m+q-1)^2} \frac{2}{\alpha} \right]^{\frac{1}{1-\alpha}} \|U_0\|_1^{1 + \frac{2q-2}{m+q-2+2/d}},
$$

where $\alpha = \frac{2q}{m+q-2+2/d}$, $\alpha_0 = \frac{2(q-1)}{m+q-2+2/d}$.

**Proof.** **Step 1.** (Uniform $L^p$ estimates for $0 < m < 2 - 2/d$) Firstly it’s obtained by multiplying the equation (1.1) with $pu$ and integrating it in $\mathbb{R}^d$ to arrive at

$$
\frac{d}{dt} \int u^p dx + \frac{4mp(p-1)}{(m+p-1)^2} \int |\nabla u^{(m+p-1)/2}|^2 dx = (p-1) \int u^{p+1} dx \leq (p-1) S_d^{-1} \|\nabla u^{(m+p-1)/2}\|_p^2 \|u\|_p^{2-m},
$$

where the last inequality (4.4) follows from (2.4) with $q = p$. Hence one has

$$
\frac{d}{dt} \int u^p dx + S_d^{-1}(p-1) \left(C^{2-m}_{d,m} - \|u\|_p^{2-m}\right) \int |\nabla u^{(m+p-1)/2}|^2 dx \leq 0.
$$

Since $\|U_0\|_p < C_{d,m}$, so the following estimate holds true for any $t > 0$

$$
\|u(\cdot, t)\|_p < \|U_0\|_p < C_{d,m},
$$

**Step 2.** (Finite time extinction for $0 < m < 1 - 2/d$) It follows from (2.3) by using $\|u\|_1 \leq \|U_0\|_1$ that

$$
\frac{\left(\|u\|_p^p\right)^{\frac{1}{p-1}} + \frac{m+2/d}{p-1} \|U_0\|_{p-1}^{\frac{1}{p-1}}}{S_d^{-1} \|U_0\|_1^{\frac{1}{p-1}}} \leq \|\nabla u\|_2^{\frac{m+2/d}{p-1}}.
$$

Substituting (4.7) into (4.4) arrives at

$$
\frac{d}{dt} \int u^p dx + \frac{(p-1)\eta}{\|U_0\|_{p-1}^{\frac{1}{p-1}}} \left(\int u^p dx\right)^{-\delta} \leq 0,
$$

where $\delta = 1 + \frac{m+2/d}{p-1} < 1$ for $m < 1 - 2/d$. Hence in view of Lemma 2.6 (iii), there exists a finite time $0 < T_{ext} \leq \frac{(\|U_0\|_p)^{-\delta}}{\eta (1-\delta)}$ with $0 < \delta = 1 + \frac{m+2/d}{p-1} < 1$ such that $\|u(\cdot, t)\|_p$ will vanish a.e. in $\mathbb{R}^d$ for all $t > T_{ext}$, thus the solution will extinct at finite time.

**Step 3.** (Uniform $L^\infty$ estimate with $r_0 := p + \epsilon_0$ for $\epsilon_0$ small enough for $1 - 2/d \leq m < 2 - 2/d$) Using (2.4) with $q = r_0$ deduces

$$
\frac{d}{dt} \int u^{r_0} dx + \frac{4mr_0(r_0-1)}{(r_0 + m - 1)^2} \int |\nabla u^{(r_0+m-1)/2}|^2 dx = (r_0-1) \int u^{r_0+1} dx
$$

$$
\leq (r_0-1) \frac{1}{S_d^{-1}} |\nabla u|^{(r_0+m-1)/2} \|u\|_{p}^{2-m} \|U_0\|_{p}^{2-m}.
$$
The last inequality is derived from (4.6). If we choose \( \epsilon_0 \) such that
\[
\eta \frac{2}{m} := \frac{4m(p + \epsilon_0)}{(p + \epsilon_0 + m - 1)^2 S_{d-1}} - \|U_0\|^{-m} < \eta,
\]
then one has
\[
\frac{d}{dt} \int u^{\eta_0} dx + S_{d-1}^{-1}(r_0 - 1) \eta \frac{2}{m} \int |\nabla u^{(m + \eta_0 - 1)/2}|^2 dx \leq 0,
\]
then we obtain the uniform estimates for \( \|u\|_{r_0} \)
\[
\|u(\cdot, t)\|_{r_0} \leq \|U_0\|_{r_0}.
\]

**Step 4.** (Uniform \( L^q \) estimates for \( q > r_0 \) with \( U_0 \in L^q(\mathbb{R}^d) \) and \( 1 - 2/d \leq m < 2 - 2/d \)) For \( q > r_0 \), taking \( r = r_0 \) in (2.5) and using (4.12) one has
\[
\frac{d}{dt} \|u\|^q_t + \frac{4qm(q - 1)}{(q + m - 1)^2} \|\nabla u^{\frac{q + m - 1}{2}}\|^2_t = (q - 1) \int u^{q+1} dx \leq \frac{2mq(q - 1)}{(m + q - 1)^2} \|\nabla u^{\frac{q + m - 1}{2}}\|^2 + C(q, r_0, d) ||u||_{r_0}^\delta,
\]
where \( \delta = 1 + \frac{1 + q - r_0}{r_0 - p} \).

Collecting (2.3) and (4.13) yields
\[
\frac{d}{dt} \|u\|^q_t \leq -\frac{2mq(q - 1)}{S_{d-1}^{-1}(m + q - 1)^2\|U_0\|^{-1/(1 + \frac{2q + d}{q - 1})}} \left( \|u\|^q_t \right)^{1 + \frac{m - 1 + 2/d}{q - 1}} + C(q, r_0, d) \|U_0\|_{r_0}^\delta.
\]

From Lemma 2.6 by letting
\[
y(t) = \|u\|^q_t, \alpha = C(q, r_0, d) \|U_0\|_{r_0}^\delta, \quad \beta = \frac{2mq(q - 1)}{S_{d-1}^{-1}(m + q - 1)^2\|U_0\|^{-1/(1 + \frac{2q + d}{q - 1})}},
\]
\[
\text{Case 1.} \ (1 - 2/d < m < 2 - 2/d) \quad a = 1 + \frac{m - 1 + 2/d}{q - 1} > 1, \text{ by (2.10) of Lemma 2.6 we have}
\]
\[
\|u\|^q_t \leq \max \left( \|U_0\|^q_0, C(\|U_0\|_1, q) \left( \|U_0\|^q_{r_0} \right)^{\frac{a}{q-1}} \right),
\]
where we have used the interpolation inequality in the last inequality for \( 1 < p < r_0 < q \).

\[
\text{Case 2.} \ (m = 1 - 2/d) \quad a = 1, \text{ from Lemma 2.6 one has}
\]
\[
y(t) \leq \alpha/\beta + y(0),
\]
where
\[
\|u(\cdot, t)\|^q_t \leq \|U_0\|^q_0 + C(\|U_0\|_1, q) \left( \|U_0\|^q_0 \right)^{\frac{a}{q-1} + \frac{q - p + 1}{q - p - 1}}.
\]

Thus we conclude that for \( m \geq 1 - 2/d \)
\[
\|u\|^q_t \leq \|U_0\|^q_0 + C(\|U_0\|_1, q) \left( \|U_0\|^q_0 \right)^{\frac{a}{q-1} + \frac{q - p + 1}{q - p - 1}}.
\]
\textbf{Step 5.} (Uniform $L^q$ estimates with $U_0 \in L^q(\mathbb{R}^d)$ and $m > 2 - 2/d$) Taking Lemma 2.3 into account we have the following estimates

$$\frac{d}{dt}\|u\|_q + \frac{4mq}{(m + q - 1)^2} \|\nabla u\|_2^{\frac{m+q-1}{2}} = (q - 1)\|u\|^{q+1}_{q+1}$$

$$\leq \frac{2mq(q - 1)}{(m + q - 1)^2} \|\nabla u\|_2^{\frac{m+q-1}{2}}$$

$$+ (q - 1) \left(1 - \frac{\alpha}{2}\right) \left[S_d \frac{2mq}{(m + q - 1)^2} \frac{2}{\alpha} \right]^{\frac{1}{\gamma-\alpha}} \|u\|_1^{1 + \frac{2(q - 1)}{\gamma}}.$$  

Here $\alpha = \frac{2q}{m+q-2+4/r}$, combining Lemma 2.2 leads to

$$\frac{d}{dt}\|u\|_q \leq -\|u\|_q + \left(1 - \frac{\alpha_0}{2}\right) \left[S_d \frac{2mq(q - 1)}{(m + q - 1)^2} \frac{2}{\alpha_0} \right]^{\frac{1}{\gamma}} \|u\|_1^{1 + \frac{2(q - 1)}{\gamma}}$$

$$+ (q - 1) \left(1 - \frac{\alpha}{2}\right) \left[S_d \frac{2mq}{(m + q - 1)^2} \frac{2}{\alpha} \right]^{\frac{1}{\gamma-\alpha}} \|u\|_1^{1 + \frac{2(q - 1)}{\gamma}},$$

where $\alpha_0 = \frac{2(q - 1)}{m+q-2+4/r}$. By Gronwall's inequality we obtain the conclusion.

As to the regularity process and global existence, we can refer to [3] for precise results. Thus ends the proof. □

The following lemma is proved by the spirit of [1] which will be used to estimate the boundedness in $L^\infty(\mathbb{R}^d)$.

\textbf{Lemma 4.1.} Assume $y_k(t) \geq 0$, $k = 0, 1, 2, \ldots$ are $C^1$ functions for $t > 0$ satisfying

$$y_k(t) \leq -y_k + a_k \left(y_{k-1}^2(t) + y_{k-1}^2(t)\right),$$  \hspace{1cm} (4.17)

where $a_k = a_3^r > 1$ with $a, r$ are positive bounded constants and $0 < \gamma_2 < \gamma_1 \leq 3$. Assume also that there exists a bounded constant $K \geq 1$ such that $y_k(0) \leq K^{3^k}$, then

$$y_k(t) \leq (2\bar{a})^{3^{k-1}} 3^{r \left(\frac{1}{3} - \frac{r}{3} - 1\right)} \max \left\{\sup_{t \geq 0} y_0^3(t), K^{3^k}\right\}. \hspace{1cm} (4.18)$$

\textbf{Proof.} Multiplying $e^t$ to both sides of (4.17) yields

$$(e^t y_k(t))' \leq a_k e^t \left(y_{k-1}^3(t) + y_{k-1}^2(t)\right) \leq 2a_k e^t \max \left\{1, \sup_{t \geq 0} y_{k-1}^3(t)\right\},$$

$$y_k(t) \leq (1 - e^{-t}) 2a_k \max \left\{1, \sup_{t \geq 0} y_{k-1}^3(t)\right\} + e^{-t} y_k(0) \hspace{1cm} (4.19)$$

$$\leq 2a_k \max \left\{1, \sup_{t \geq 0} y_{k-1}^3(t), y_k(0)\right\}$$

$$\leq 2a_k \max \left\{1, \sup_{t \geq 0} y_{k-1}^3(t), K^{3^k}\right\} = 2a_k \max \left\{\sup_{t \geq 0} y_{k-1}^3(t), K^{3^k}\right\}.$$
Then from (4.19) after some iterative steps we have
\[ y_k(t) \leq 2a_k(2a_{k-1})^3(2a_{k-2})^3 \cdots (2a_1)^3 \max_{t \geq 0} \left\{ \sup_{t \geq 0} y_0^{3k}(t), C^{3k} \right\} \]
\[ = (2a)^{1+3+3^2+3^3+\cdots+3^{k-1}} (k+3(k-1)+3^2(k-2)+\cdots+3^{k-1}) \max_{t \geq 0} \left\{ \sup_{t \geq 0} y_0^{3k}(t), C^{3k} \right\} \]
\[ = (2a)^{\frac{3^{k-1}}{2^2}} 3^r (\frac{3^{k-1}}{2^2} - \frac{3}{2}) \max_{t \geq 0} \left\{ \sup_{t \geq 0} y_0^{3k}(t), C^{3k} \right\}. \]

Now we are in a position to prove the $L^\infty$ bound.

**Theorem 4.2.** Let $d \geq 3$, $m > 0$. Assume $U_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$. For $0 < m < 2 - 2/d$, we also assume $\|U_0\|_p < C_{d,m}$. Then there exists a weak solution of (1.1) such that for any $t > 0$
\[ \|u\|_{L^\infty} \leq C(m, d, K_0), \]
where $K_0 = \max\{1, \|U_0\|_1, \|U_0\|_\infty\}$. Furthermore, if $\nabla U_0^m \in L^2(\mathbb{R}^d)$, then for any $T > 0$, the weak solution has the following regularities
\[ u(x, t) \in L^\infty(0, T; L^1_+ \cap L^\infty(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d)), \]
and
\[ \nabla u \in L^2(0, T; H^{-1}(\mathbb{R}^d)), \quad (u^{\frac{m+1}{m}})_t \in L^2(0, T; L^2(\mathbb{R}^d)). \]

**Proof of Theorem 4.2.** The global existence of the weak solution has been proved in Theorem 2.17 of [3] with $U_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$. Now we will focus on the boundedness in $L^\infty(\mathbb{R}^d)$ uniformly in time. Firstly we denote $q_k = 3^k + m + d + 1$ and estimate $\int_{\mathbb{R}^d} u^{q_k} \, dx$.

**Step 1.** (The $L^{q_k}$ estimate) Similar to the proof from (3.6) to (3.10) of Theorem 1.1, we also obtain
\[ \frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} \, dx \leq -C_2 \int_{\mathbb{R}^d} \left| \nabla u^{\frac{q_k+1}{2}} \right|^2 \, dx \]
\[ + C(\delta_1)(q_k - 1)C_1^{\frac{1}{\gamma_1-1}} \left( \int_{\mathbb{R}^d} u^{q_k-1} \, dx \right)^{\gamma_1}. \]

Here $C_2 = m/2$, $C_1 = m/2(q_k-1)$ and
\[ \gamma_1 = 1 + \frac{2b - 2a}{2d - bd + 2a} \leq 3, \quad \delta_1 = 2 \left( \frac{1}{a} - \frac{d-2}{2d} \right) = O(1), \]
where
\[ \tilde{a} = \frac{2q_k - 2a}{q_k + m - 1}, \quad \tilde{b} = \frac{2(q_k + 1)}{q_k + m - 1}. \]

Moreover, taking
\[ a = \frac{2q_k - 2a}{q_k + m}, \quad b = \frac{2q_k - 2a}{q_k + m}, \quad w = u^{\frac{q_k+1}{2}}. \]
in Lemma 2.1 we have
\[
\int_{\mathbb{R}^d} u^{q_k} \, dx \leq C(\delta_2)C_2^{-\frac{1}{r-1}} \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_2} + C_2 \left\| \nabla u^{\frac{q_k + m - 1}{2}} \right\|^2_2,
\]
(4.24)
where \( \delta_2 = \frac{2 \left( \frac{1}{r} - \frac{d-2}{2} \right)}{\frac{1}{r} - 1} = O(1) \) and \( \gamma_2 = 1 + \frac{2b-2a}{2d-4a+2b} \leq 3 \) if \( m > 0 \).

Plugging (4.24) into (4.23) one has
\[
\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} \, dx
\leq - \int_{\mathbb{R}^d} u^{q_k} \, dx + C(\delta_1)(q_k - 1)C_1^{-\frac{1}{r-1}} \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_1}
+ C(\delta_2)C_2^{-\frac{1}{r-1}} \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_2}
= - \int_{\mathbb{R}^d} u^{q_k} \, dx + C(\delta_1, m)(q_k - 1)^{\frac{1}{r-1}} \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_1}
+ C(\delta_2, m) \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_2}
\leq - \int_{\mathbb{R}^d} u^{q_k} \, dx
+ \max[1, C(\delta_1, m), C(\delta_2, m)] q_k^{\frac{1}{r-1}} \left\{ \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_1} + \left( \int_{\mathbb{R}^d} u^{q_{k-1}} \, dx \right)^{\gamma_2} \right\},
\]
where \( \gamma_2 < \gamma_1 \leq 3 \) with \( m > 0 \).

**Step 2.** (Uniform estimates of \( L^\infty(\mathbb{R}^d) \)) Let \( K_0 = \max(1, \|U_0\|_1, \|U_0\|_\infty) \) and \( K = K_0^{\frac{m}{q_k}} \geq 1 \), then
\[
y_k(0) = \|U_0\|_{q_k} = \left[ \max(\|U_0\|_1, \|U_0\|_\infty) \right]^{q_k} \leq K_0^{q_k} = K_0^{3^k}.
\]
(4.26)

Take
\[
y_k(t) = \int_{\mathbb{R}^d} u^{q_k} \, dx, \quad r = \frac{1}{1 - 1/\delta_1},
\]
\[
\bar{a} = \max[1, C(\delta_1, m), C(\delta_2, m)] (m + d + 1)^r = O(1),
\]
then (4.25) can be recast as
\[
y_k'(t) \leq -y_k(t) + \bar{a}3^k \left( y_{k-1}^1(t) + y_{k-1}^2(t) \right).
\]
(4.27)

Combining (4.26) and (4.27), by Lemma 4.1 we obtain
\[
\int_{\mathbb{R}^d} u^{q_k} \, dx \leq (2\bar{a})^{\frac{3^k}{2^{k+1} - 3^k}} 3^r \left( \frac{3^k + 1}{3^k - 2} - \frac{1}{2} \right) \max \left\{ \sup_{t \geq 0} y_0^k(t), K_0^{3^k} \right\}.
\]
(4.28)

Recalling \( q_k = 3^k + m + d + 1 \) and taking the power \( \frac{1}{q_k} \) to both sides of (4.28), then the boundedness of the solution \( u \) is obtained by passing to the limit \( k \to \infty \)
\[
\|u(t)\|_{L^\infty} \leq \sqrt{2\bar{a}} 3^{r/2} \max \left( \sup_{t \geq 0} y_0(t), K_0 \right).
\]
(4.29)

Now we shall divide it into two cases \( m > 2 - 2/d \) and \( 0 < m < 2 - 2/d \) to estimate \( y_0(t) \).
Case 1. \((m > 2 - 2/d)\) Thanks to Proposition 1, taking \(q = m + d + 2\) in (4.3) and using the interpolation inequality by \(U_0 \in L^1 \cap L^\infty (\mathbb{R}^d)\) we have
\[
\|u(t)\|_{m+2}^2 \leq \|U_0\|_{m+2+2}^2 + C(m, d, \|U_0\|_1) \leq K_0^{m+2+2} + C(m, d, \|U_0\|),
\]
where \(K_0 = \max\{1, \|U_0\|_1, \|U_0\|\}\). Hence from (4.29) one has
\[
\|u(t)\|_{L^\infty} \leq \sqrt{\frac{3^{3/4}}{2\pi \alpha}} \max_{t \geq 0} (\|u(t)\|_{m+2+2}^{m+2+2}, K_0)
\leq \sqrt{\frac{3^{3/4}}{2\pi \alpha}} \max (\|u(t)\|_{m+2+2}^{m+2+2}, K_0).
\]

Case 2. \((0 < m \leq 2 - 2/d)\) For \(0 < m \leq 2 - 2/d\), it’s easy to verify \(m + 2 > p\), therefore by (4.1) of Proposition 1 we have
\[
\|u\|_{m+2}^2 \leq C(||U_0||_1, m, d) \left(\|U_0\|_{m+2+2}^{p+\epsilon_0-1} \frac{m+2+2-p+\epsilon_0}{m+2+2-p+\epsilon_0} \right) + \|U_0\|_{m+2}^2.
\]
Thus from (4.29) one has
\[
\|u(t)\|_{L^\infty} \leq \sqrt{\frac{3^{3/4}}{2\pi \alpha}} \max (\|u(t)\|_{m+2+2}^{m+2+2}, K_0)
\leq \sqrt{\frac{3^{3/4}}{2\pi \alpha}} \left(\|U_0\|_1, m, d \right) \left(K_0^{m+2+2} \frac{\epsilon_0^{p+\epsilon_0-1} \frac{m+2+2-p+\epsilon_0}{m+2+2-p+\epsilon_0} + K_0^{m+2+2}}\right),
\]
where \(\epsilon_0\) satisfies
\[
\frac{4m(p+\epsilon_0)}{(p+\epsilon_0 + m - 1)^2} - \|U_0\|_p^{2-m} = \frac{\eta_0}{2}.
\]

Step 3. (Time regularity for \(m > 1 - 2/d\)) It directly follows from \(u(x, t) \in L^\infty (0, T; L^1 \cap L^\infty (\mathbb{R}^d))\) that
\[
\|\nabla u\|_{L^2(0, T; L^2(\mathbb{R}^d))} \leq C,
\]
\[
\|u\nabla c\|_{L^\infty (0, T; L^\infty(\mathbb{R}^d))} \leq C,
\]
\[
\|\nabla u^m\|_{L^2(0, T; L^2(\mathbb{R}^d))} \leq C,
\]
these computations can derive the time regularities (4.20). Furthermore, Multiplying \(\frac{\partial u^m}{\partial t}\) to both sides of (1.1) we obtain
\[
\frac{4m}{(m+1)^2} \int_{\mathbb{R}^d} \left|\left(\frac{u^m}{t} \right)_t\right|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u^m|^2 dx
= -m \int_{\mathbb{R}^d} \nabla u \cdot \nabla u^m - u_m^m dx + m \int_{\mathbb{R}^d} u^m u_t dx
= - \frac{2m}{m+1} \int_{\mathbb{R}^d} u^m \nabla u \cdot \nabla cdx + m \int_{\mathbb{R}^d} u^m u_t dx
\leq - \frac{2m}{(m+1)^2} \int_{\mathbb{R}^d} \left|\left(\frac{u^m}{t} \right)_t\right|^2 dx + C(m) \int_{\mathbb{R}^d} |u^m \nabla u \cdot \nabla c|^2 dx + C(m) \int_{\mathbb{R}^d} u^m dx.
Hence for any $t > 0$, from $\int_{\mathbb{R}^d} u^{m+3} dx \leq C(\|U_0\|_{m+3}, d, m)$ one has
\[
\frac{2m}{(m+1)^2} \int_0^t \int_{\mathbb{R}^d} \left| \left( u^{\frac{m+1}{2}} \right)_t \right|^2 dx ds + \frac{1}{2} \int_{\mathbb{R}^d} \| \nabla u(t) \|^2 dx \tag{4.32}
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^d} \| \nabla U_0 \|^2 dx + C(m) \int_0^t \int_{\mathbb{R}^d} \left| \nabla u^{\frac{m+1}{m}} \cdot \nabla c \right|^2 dx ds + C(\|U_0\|_{m+3}, d, m)
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^d} \| \nabla U_0 \|^2 dx + C(m) \| \nabla c \|^2_{L^\infty(0, t; L^\infty(\mathbb{R}^d))} \int_0^t \int_{\mathbb{R}^d} \left| \nabla u^{\frac{m+1}{m}} \right|^2 dx ds + C(\|U_0\|_{m+3}, d, m).
\]

It follows from the Young inequality that
\[
\| \nabla c \|_{L^\infty(\mathbb{R}^d)} = C(d) \left\| u(x) \ast \frac{1}{|x|^{d-1}} \right\|_{L^\infty(\mathbb{R}^d)} = C(d) \left[ \int_{0<|x-y| \leq 1} \frac{u(y)}{|x-y|^{d-1}} dy + \int_{|x-y| > 1} \frac{u(y)}{|x-y|^{d-1}} dy \right]_{L^\infty(\mathbb{R}^d)}
\]
\[
\leq C(d) \left( \| u(y) \|_{L^\infty(\mathbb{R}^d)} \left\| \frac{1}{|x|^{d-1}} \right\|_{L^1(0<|x| \leq 1)} + \| u \|_{L^1(\mathbb{R}^d)} \right)
\]
\[
\leq C(d) \left( \| u \|_{L^\infty(\mathbb{R}^d)} + \| u \|_{L^1(\mathbb{R}^d)} \right),
\]
and the initial data $U_0 \in L^2(\mathbb{R}^d)$ leads to
\[
\int_0^t \int_{\mathbb{R}^d} \left| \nabla u^{\frac{m+1}{m}} \right|^2 dx ds \leq C(\|U_0\|_2, d, m). \tag{4.34}
\]

Plugging (4.33) and (4.34) into (4.32) we obtain the time regularities (4.21) and (4.22). Thus completes the proof. \qed

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