Section 2.1 First Order Linear Differential Equations

1. Determine the order of the following equations. Are they linear or nonlinear?

$(a)\ \ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + t^3 y^5 = 0$

$(b)\ \ \sin(t^3)\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} + \cos(t^3)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + ty = t^2 e^t$

$(c)\ \ y\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y^2\frac{\mathrm{d}y}{\mathrm{d}t} + t = 0$

Definition: The **order** of a differential equation is the highest order of the derivatives in the equation.

Definition: A **linear** ordinary differential equation takes the form of

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_0(t)y = g(t)$$

Comment: There are two classes of differential equations. Ordinary differential equations (ODE) contain only ordinary derivatives, while partial differential equations (PDE) contain partial derivatives.

General form of a first order linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t) \tag{1}$$

2. Consider the equation $y' + \frac{1}{2}y = 2 + t$.
(a) Is this a first order linear differential equation? If yes, $p(t) =$? $g(t) =$?

(b) Find $\mu(t) = \exp \int p(t)dt$.
Definition: $\mu(t)$ is an **integrating factor**.
Comment: You can simply drop the constant C from the integrating factor.

(c) Multiply both sides of the equation by $\mu(t)$.

(d) Rewrite the left-hand side of the equation as the derivative of some expression.

(e) Find the general solution in explicit form.
Comment: **Explicit form** means y is isolated.

A systematical way to solve equation (1) is as follows.

Step 1: Multiply both sides of the equation by $\mu(t) = \exp \int p(t)dt$.

Step 2: Rewrite the left side as $\frac{d}{dt}(\mu y)$.

Step 3: Integrate both sides.

3. Justify the formula $\mu(t) = \exp \int p(t)dt$.

4. Find the general solution in explicit form for the equation $\frac{dy}{dt} = ay + b$, where a, b are constants. Discuss the behavior of y as $t \to \infty$.

5. Solve the initial value problem

$$\begin{aligned} ty' + 2y &= 4t^2 \\ y(1) &= 2 \end{aligned}$$

Comment: Always write the differential equation in the standard form (1) before calculating the integrating factor.