Nonexistence of minimizer for Thomas-Fermi-Dirac-von Weizsäcker model

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1 Introduction

In this paper, we study the following energy functional

$$E(\phi) := \int_{\mathbb{R}^3} |\nabla \phi|^2 + F(\phi^2) \, dx + D(\phi^2, \phi^2),$$

where $F(t) = t^{5/3} - t^{4/3}$ and $D(\cdot, \cdot)$ is the Coulomb interaction in $\mathbb{R}^3$:

$$D(f, g) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(x) g(y) \frac{1}{|x - y|} \, dx \, dy.$$

Our main result is the following nonexistence result.

**Theorem 1.** There exists constant $m_0 > 0$, such that the variational problem

$$\inf_{\phi \in H^1(\mathbb{R}^3), \int \phi^2 = m} E(\phi)$$

does not have a minimizer if $m \geq m_0$.

We also consider a “sharp interface” version of the functional (1.1). For three-dimensional sets $\Omega \subset \mathbb{R}^3$, define

$$E(\Omega) := |\partial \Omega| + \int_{\Omega \times \Omega} \frac{1}{|x - y|} \, dx \, dy.$$  

A nonexistence result similar to Theorem 1 also holds.

**Theorem 2.** There exists constant $V_0 > 0$, such that the variational problem

$$\inf_{|\Omega| = V} E(\Omega)$$

does not have a minimizer if $V \geq V_0$.

If the Coulomb term is not present in (1.1) and (1.3), a standard Modica-Mortola type result establishes a link between the two functional through Gamma convergence when the volume constraint $m$ is large [19, 20]. This is no longer valid for the energy functionals (1.1) and (1.3) due to the Coulomb term, which breaks the...
similarity of the solutions on large length scale. The connection between the two functionals is just on the formal level.

We will prove Theorem 1 and 2 separately. While our proof for Theorem 1 can also be used to prove Theorem 2 after straightforward modifications, it is not clear how to extend our proof for Theorem 2 to the “diffuse interface” case (1.1).

The rest of the article is organized as follows. After discussion of motivations and some remarks, we first present the proof of Theorem 2 in Section 2. The proof of Theorem 1 follows in Section 3.

In the physics literature, the functional (1.1) is known as the Thomas-Fermi-Dirac-von Weizsäcker (TFDW) model for electrons with no external potential, where \( \rho = \varphi^2 \) is the electron density. A mathematical introduction to the model and its physical motivation can be found in [13–15] and references therein. In particular, the existence of minimizer for small \( m \) was proved in [18, Corollary II.2]. It is also related with the Schrödinger-Poisson-Slater system, see for example [23, 24].

The sharp interface functional (1.3) recently receives attention from many authors, see for example [1, 4, 5, 7, 8, 12, 22, 29]. It is a natural extension of the isoperimetric problem. It is connected to various mean-field type models such as Ohta-Kawasaki, Ginzburg-Landau, and Thomas-Fermi type models. The existence of minimizer for small \( V \) follows from results in [5]. The version of Theorem 2 in two space dimensions has recently been proved in [12].

The nonexistence problem of (1.2) arises when we consider the small volume-fraction limit for electrons in a box with uniform background charge (physically known as the Jellium model). Using physical arguments, one expects the formation of crystalline structures made of separated electron “bubbles”, known as Wigner crystallization [30]. The proof of Wigner crystallization is an open challenging problem. If the TFDW model is used, the energy functional (1.1) then describes the self-energy of such a single electron “bubble”. Theorem 1 states that such a bubble can only contain a finite amount of electrons, and hence, the system will indeed break up into many bubbles in the limit. We hope the result here can be used to understand Wigner crystallization in TFDW model, which is still open.

Related to the Wigner crystallization, the small volume-fraction limit for an Ohta-Kawasaki type model was studied in [5, 6], where they assume Theorem 2 as part of their conjecture in [6, Section 6] to make sure that multiple bubbles arise. See also small volume-fraction limits for related models studied recently in [11, 21, 25].

Physically, Theorem 1 can also be interpreted as “the vacuum can only bind a finite number of electrons” in the TFDW model. This is related to the ionization conjecture in quantum mechanics, which states that the number of electrons that can be bound to an atomic nucleus of charge \( Z \) cannot exceed \( Z + 1 \). The ionization conjecture, while still open for the Schrödinger equation, has been studied by many authors for different types of models in quantum mechanics (see e.g. [2, 3, 9, 16, 17, 26–28]). In particular, for the Thomas-Fermi-von Weizsäcker model, where the
non-convex term in the TFDW model $\int -\varphi^{8/3}$ is dropped, the ionization conjecture was proved in [3]. For the TFDW model, however, it is still open whether an atom nucleus can only bind a finite number of electrons. Theorem 1 confirms this for $Z = 0$ (system without atoms). Our proof for Theorem 1 however does not cover the case with external potential caused by an atomic nucleus, since we have used translational invariance in the proof. The case with atoms will be investigated in future work.

2 Isoperimetric problem with nonlocal interaction

We prove Theorem 2 in this section. We start the proof with the following lemma collects two elementary properties of the variational problem (1.4).

Lemma 3. Let $V \gg 1$.

(i) We have

$$E(V) := \inf \{ E(\Omega) \mid |\Omega| = V \} \lesssim V.$$

(ii) Let $\Omega$ be a minimizer of (1.4), $\Omega$ is then connected. (Connectedness is defined in a measure-theoretic sense).

Proof. We start by establishing the inequality

$$(2.1) \quad E(V) \leq E(V_0) + E(V_1) \quad \text{for} \quad V = V_0 + V_1.$$

To prove this inequality, we first argue that $E(V)$ could have also been defined on the basis of bounded sets:

$$E(V) = \inf \{ E(\Omega) \mid |\Omega| = V, \Omega \text{ is bounded} \}.$$  

This inequality (2.1) can be seen as follows: For given $\Omega$ with $|\Omega| = V$ and $E(\Omega) < \infty$ we have to argue that there exists a bounded $\tilde{\Omega}$ with $|\tilde{\Omega}| = V$ and such that $E(\tilde{\Omega})$ is only slightly larger than $E(\Omega)$. Indeed, because of $|\Omega| < \infty$, there exists a possibly large $R < \infty$ such that $|\Omega \backslash B_R(0)| \ll 1$ and $|\partial B_R(0) \cap \Omega| \ll 1$. Consider the stretched, cut-off set $\tilde{\Omega} := \lambda (\Omega \cap B_R(0))$, where $\lambda := V / |\Omega \backslash B_R(0)| \approx 1$ is defined such that $|\tilde{\Omega}| = V$. We have for the Coulomb energy

$$\int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{1}{|x-y|} \, dx \, dy = \lambda^5 \int_{\Omega \cap B_R(0)} \int_{\Omega \cap B_R(0)} \frac{1}{|x-y|} \, dx \, dy \leq \lambda^5 \int_{\Omega} \int_{\Omega} \frac{1}{|x-y|} \, dx \, dy$$

and for the interfacial energy

$$|\partial \tilde{\Omega}| = \lambda^2 |\partial (\Omega \cap B_R(0))| \leq \lambda^2 (|\partial \Omega| + |\partial B_R(0) \cap \Omega|).$$

Hence we obtain for the total energy

$$E(\tilde{\Omega}) \leq \lambda^5 E(\Omega) + \lambda^2 |\partial B_R(0) \cap \Omega|$$

and the last expression is $\approx E(\Omega)$ since $\lambda \approx 1$ and $|\partial B_R(0) \cap \Omega| \ll 1$. 

We now turn to inequality \((2.1)\). According to \((2.2)\), for given \(\varepsilon > 0\), there exist bounded sets \(\Omega_i, i = 0, 1\), with \(E(\Omega_i) \leq E(V_i) + \varepsilon\). Let \(R < \infty\) be so large that \(\Omega_i \subset B_R(0)\). Define \(\tilde{\Omega} := \Omega_0 + (d + \Omega_1)\) with a shift vector \(d\) with \(|d| > 2R\). Note that for \(x \in \Omega_0\) and \(y \in d + \Omega_1\) we have \(|x - y| \geq d - 2R > 0\). In particular, we have \(|\partial \tilde{\Omega}| = |\partial \Omega_0| + |\partial \Omega_1|\) and \(\tilde{\Omega} = |\Omega_0| + |\Omega_1| = V_0 + V_1\), so that we may use \(\tilde{\Omega}\) in the definition of \(E(V)\):

\[
E(V) \leq E(\tilde{\Omega}) \\
\leq E(\Omega_0) + E(\Omega_1) + \frac{V_0V_1}{d - 2R} \\
\leq E(V_0) + E(V_1) + 2\varepsilon + \frac{V_0V_1}{d - 2R}.
\]

Letting first \(d\) tend to infinity and then \(\varepsilon\) to zero yields \((2.1)\).

Argument for (i). If \(\Omega\) is a ball of volume \(V\), we have \(E(\Omega) \sim V^{2/3} + V^{5/3}\). In particular, if \(\Omega\) is a ball of volume \(V \sim 1\), we have \(E(\Omega) \sim 1\). We reformulate this as

\[
E(V) \lesssim 1 \quad \text{for } 1 \leq V < 2.
\]

Using the subadditivity \((2.1)\), we obtain by induction in \(N\)

\[
E(V) \lesssim N \quad \text{for } N \leq V < N + 1.
\]

Argument for (ii). Later in the proof, we will need connectedness only in the following measure-theoretic sense: Suppose that for some ball \(B_R\) we have both \(\Omega \cap B_R \neq \emptyset\) and \(\Omega \setminus B_R \neq \emptyset\) in the following sense

\[
(2.3) \quad |\Omega \cap B_R| > 0 \quad \text{and} \quad |\Omega \setminus B_R| > 0.
\]

Then we claim that we have \(\partial B_R \cap \Omega \neq \emptyset\) in the sense of

\[
(2.4) \quad |\partial B_R \cap \Omega| > 0.
\]

(For a Caccioppoli set \(\Omega\), its characteristic function \(\chi\) admits an inner trace \(\chi^+\) on \(\partial B_R\). Thus \((2.4)\) means \(\int_{\partial B_R} \chi^+ d\mathcal{H}^2 > 0\).) We establish \((2.4)\) by contradiction. Suppose it were not true. Let us introduce \(\Omega_0 := \Omega \cap B_R\) and \(\Omega_1 := \Omega \setminus B_R\). Since by assumption \((2.3)\), \(V_0 := |\Omega_0| > 0\) and \(V_1 := |\Omega_1| > 0\) we have for the mixed Coulomb energy

\[
\int_{\Omega_0} \int_{\Omega_1} \frac{1}{|x - y|} \, dx \, dy > 0,
\]

so that we obtain for the total Coulomb energy

\[
\int_{\Omega} \int_{\Omega} \frac{1}{|x - y|} \, dx \, dy > \int_{\Omega_0} \int_{\Omega_0} \frac{1}{|x - y|} \, dx \, dy + \int_{\Omega_1} \int_{\Omega_1} \frac{1}{|x - y|} \, dx \, dy.
\]

For the interfacial energy we have because of our erroneous assumption \(|\partial B_R \cap \Omega| = 0\):

\[
|\partial \Omega| = |\partial \Omega_0| + |\partial \Omega_1|.
\]
Combining the two last inequalities, we obtain
\[ E(V) = E(\Omega) > E(\Omega_0) + E(\Omega_1) \geq E(V_0) + E(V_1), \]
which in view of the obvious \( V = V_0 + V_1 \) is in contradiction to the subadditivity (2.1).

The following — crucial — lemma amounts to an elementary regularity estimate for a minimizer \( \Omega \). In the rough terms of volume, it states that the set cannot be thinner than order one. It is based on the global a priori estimate in Lemma 3(i) and an elementary argument from the theory of minimal surfaces, see for instance [10, Proposition 5.14], that we slightly adapt to deal with the “perturbation” through the volume constraint and the Coulomb energy. These two effects are lower order perturbations in the sense that they only affect length scales of order one or larger, but do not affect the regularity on length scales small compared to one.

**Lemma 4.** Let \( \Omega \) be a minimizer of (1.4) with prescribed volume \( V \geq 1 \). Then for \( R \leq 1 \) and all \( x \in \Omega \) such that \( |\Omega \cap B_r(x)| > 0, \forall r > 0, \)
\[ |\Omega \cap B_R(x)| \gtrsim R^3. \]  

**Proof.** Due to translational invariance, we can assume without loss of generality that \( x = 0 \). It is obviously enough to show (2.5) for
\[ R \ll 1, \quad \text{in particular } R^3 \ll V. \]
We will compare \( \Omega \) to \( \tilde{\Omega} \) given by
\[ \tilde{\Omega} := \lambda (\Omega \setminus B_R), \]
where the stretching factor \( \lambda > 1 \) makes sure that \( |\tilde{\Omega}| = V \). Hence \( \lambda \) is given by
\[ \lambda^3 := \frac{V}{|\Omega \setminus B_R|} = \frac{V}{V - |\Omega \cap B_R|}. \]
We note that because of \( |\Omega \cap B_R| \lesssim R^3 \) and (2.6) we have
\[ \lambda^3 - 1 \lesssim \frac{|\Omega \cap B_R|}{V} \ll 1. \]

We now turn to a comparison of the energies: For the Coulomb energy we have
\[ \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{1}{|x - y|} \, dx \, dy \overset{(2.7)}{=} \lambda^5 \int_{\Omega - B_R} \int_{\Omega - B_R} \frac{1}{|x - y|} \, dx \, dy \leq \lambda^5 \int_{\Omega} \int_{\Omega} \frac{1}{|x - y|} \, dx \, dy. \]
For the interfacial energy we note
\[
|\partial \tilde{\Omega}| = \lambda^2 |\partial (\Omega \backslash B_R)|
\]
\[
\tag{2.7}
\leq \lambda^2 (|\partial \Omega| + |\partial B_R \cap \Omega| - |\partial \Omega \cap B_R|)
\]
\[
\leq \lambda^5 |\partial \Omega| + \lambda^2 |\partial B_R \cap \Omega| - |\partial \Omega \cap B_R|.
\]

Hence from \(E(\Omega) \leq E(\tilde{\Omega})\) we obtain
\[
|\partial \Omega \cap B_R| \leq (\lambda^5 - 1)E(\Omega) + \lambda^2 |\partial B_R \cap \Omega|,
\]
which using \(|\partial (\Omega \cap B_R)| = |\partial \Omega \cap B_R| + |\partial B_R \cap \Omega|\) we rewrite as
\[
(2.10) \quad |\partial (\Omega \cap B_R)| \leq (\lambda^5 - 1)E(\Omega) + (\lambda^2 + 1)|\partial B_R \cap \Omega|.
\]

Now convert the inequality (2.10) into a suitable estimate. We use the estimates

\begin{itemize}
  \item the isoperimetric inequality \(|\Omega \cap B_R|^{2/3} \lesssim |\partial (\Omega \cap B_R)|\),
  \item the global energy bound \(E(\Omega) \lesssim V\), cf. Lemma 3(i),
  \item the estimate
    \[
    \lambda^5 - 1 \underbrace{\lesssim}_{(2.9)} 1 - \frac{1}{\lambda^3} \underbrace{\lesssim}_{(2.8)} \frac{|\Omega \cap B_R|}{V},
    \]
  \item the estimate \(\lambda^2 \lesssim 1\) in particular following from (2.9).
\end{itemize}

We so obtain
\[
|\Omega \cap B_R|^{2/3} \lesssim |\Omega \cap B_R| + |\partial B_R \cap \Omega|.
\]

Since \(|\Omega \cap B_R| \lesssim R^3 \underbrace{\ll}_{(2.6)} 1\), we have \(|\Omega \cap B_R| \ll |\Omega \cap B_R|^{2/3}\) so that the last estimate simplifies to
\[
|\Omega \cap B_R|^{2/3} \lesssim |\partial B_R \cap \Omega|.
\]

We finally note that \(|\partial B_R \cap \Omega| = \frac{d}{dR} |\Omega \cap B_R|\) (and that \(|\Omega \cap B_R| > 0\) by assumption). Hence the last estimate turns into the differential inequality
\[
\frac{d}{dR} |\Omega \cap B_R|^{1/3} \gtrsim 1.
\]

Integrating this inequality yields the desired lower bound.

Together with Lemma 3(i), the next lemma yields Theorem 2 as a corollary. It combines Lemma 3(ii) with Lemma 4 in an elementary way:

**Lemma 5.** Let \(\Omega\) be a minimizer of prescribed volume \(V \gg 1\). Then we have for the Coulomb energy
\[
(2.11) \quad \int_{\Omega} \int_{\Omega} \frac{1}{|x-y|} \, dx \, dy \gtrsim V \ln V.
\]
**Proof.** We will first argue that Lemma 3(ii) and Lemma 4 imply that \( \Omega \) cannot be too thin in the sense of

\[
\Omega \cap B_R(x) \gtrsim R \quad \text{for all points } x \in \Omega \text{ and radii } 1 < R \leq \frac{1}{2} \text{diam}(\Omega).
\]

Indeed, fix a point \( x \in \Omega \) and a radius with \( 1 < R \leq \frac{1}{2} \text{diam}(\Omega) \). Because of the latter, \( \Omega \) is not contained in \( B_R(x) \). According to the connectedness of \( \Omega \) as in Lemma 3(ii), there exist points \( x_1, x_2, \ldots, x_{\lfloor R \rfloor} \) such that

\[
\Omega \cap \partial B_R(x_i) \quad \text{for } i = 1, 2, \ldots, \lfloor R \rfloor.
\]

(Indeed, we have for the inner trace \( \chi_i \) of the characteristic function \( \chi \) of \( \Omega \) that \( \int_{\partial B_{R-i}} \chi_i \, d\mathcal{H}^2 > 0 \). Take a Lebesgue point \( x_i \) of \( \chi_i \) with \( \chi_i(x_i) = 1 \). It will satisfy \( x_i \in \Omega \cap \partial B_R(x_i) \) and \( |\Omega \cap B_r(x_i)| > 0 \) for any \( r > 0 \).) Here \( \lfloor R \rfloor \) is the largest integer less than \( R \). It is clear from (2.13) that the balls \( \{B_{1/2}(x_i)\}_{i=1, \ldots, \lfloor R \rfloor} \) are pairwise disjoint. According to Lemma 4 we have

\[
|\Omega \cap B_R(x)| \geq \sum_{i=1}^{\lfloor R \rfloor} |\Omega \cap B_{1/2}(x_i)| \gtrsim |R|.
\]

This yields (2.12).

We now argue that (2.12) implies

\[
\int_\Omega \int_\Omega \frac{1}{|x-y|} \, dx \, dy \gtrsim V \text{diam}(\Omega).
\]

The desired estimate (2.11) then follows from this estimate combined with \( \text{diam} \Omega \gtrsim V^{1/3} \). We first note that for any \( 1 \leq R \leq \frac{1}{2} \text{diam}(\Omega) \) we have for the six-dimensional volume

\[
|\{ (x,y) \in \Omega \times \Omega \mid |x-y| < R \}| = \int_\Omega |\{ y \in \Omega \mid |x-y| < R \}| \, dx
\]

\[
= \int_\Omega |\Omega \cap B_R(x)| \, dx \overset{(2.12)}{\gtrsim} R |\Omega|.
\]

Hence we have as desired

\[
\int_\Omega \int_\Omega \frac{1}{|x-y|} \, dx \, dy = \int_0^{\infty} \frac{1}{R} \left| \partial \{ (x,y) \in \Omega \times \Omega \mid |x-y| < R \} \right| \, dR
\]

\[
\overset{\text{diam}(\Omega)}{=} \int_0^{\infty} \frac{1}{R} \, dR \left| \{ (x,y) \in \Omega \times \Omega \mid |x-y| < R \} \right| \, dR
\]

\[
\gtrsim \int_1^{\frac{1}{2} \text{diam}(\Omega)} \frac{1}{R} \, dR |\Omega|.
\]

\( \square \)
3 Thomas-Fermi-Dirac-von Weizsäcker model

We prove Theorem 1 in this section. Assume the minimizer of (1.2) exists, denote it by ϕ. We first note an elementary property of ϕ:

**Lemma 6.** We have ϕ(x) ∈ [0, (4/5)^3/2] and −ϕ^2(x) ≤ F(ϕ^2(x)) ≤ 0 for a.e. x ∈ R^3.

**Proof.** Since E(|φ|) ≤ E(φ), we may restrict to φ ≥ 0. Note that argmin_{t≥0} F(t) = (4/5)^3. Take φ_1 = φ ∧ (4/5)^3/2 and let m_1 = ∫ φ_1^2 ≤ m. Since φ minimizes (1.2), we have

\[ E(φ) ≤ E(φ_1) + \inf_{ψ^2 = m - m_1} E(ψ). \]

Indeed, for any ψ with ∫ ψ^2 = m − m_1, let

ψ_L(x) = Z_L[φ_1(x) + ψ(x + Le_1)],

where e_1 = (1, 0, 0) ∈ R^3 and Z_L is a normalization constant so that ∫ ψ^2 = ∫ φ^2 = m. Taking L → ∞, it is clear that Z_L → 1 and

\[ \limsup_{L→∞} E(ψ_L) = E(φ_1) + E(ψ). \]

We arrive at (3.1) as E(φ) ≤ E(ψ_L) for any L.

Consider the scaling ψ_η = η^{-3/2} ψ(x/η), so ∫ ψ_η^2 = ∫ ψ^2. It is not hard to see that E(ψ_η) → 0 as η → ∞, as each term in (1.1) goes to zero. Therefore, we have E(φ) ≤ E(φ_1) as

\[ \inf_{ψ^2 = m - m_1} E(ψ) ≤ 0. \]

Suppose \(|\{φ > (4/5)^3/2\}| > 0\), then as (4/5)^3 is the unique minimizer of F(t) for t ≥ 0,

\[ ∫ F(φ_1^2) < ∫ F(φ^2). \]

Note that |∇φ_1| ≤ |∇φ| and D(φ_1^2, φ_1^2) ≤ D(φ^2, φ^2), we arrive at a contradiction E(φ_1) < E(φ). Therefore, φ ≤ (4/5)^3/2. The remainder of the lemma follows easily. □

The next lemma establishes a key estimate of the minimizer.

**Lemma 7.** For any x ∈ R^3 and two radii r, R > 0 such that R > r + 1, we have

\[ ∫_{B_r(x) \setminus B_r(1)} φ^2(y) dy ≥ \frac{1}{R + r} ∫_{B_r(x)} φ^2(y) dy \int_{B_R(x) \setminus B_{r+1}(x)} φ^2(y) dy. \]

**Proof.** By translational invariance, it suffices to prove the lemma for x = 0. To prove the lemma, we will compare the energy of φ with a competitor obtained by cutting and moving part of the minimizer to infinity.

Let us take two functions f_1, f_2 : R → [0, 1] satisfying the following properties:
(i) \( f_1(t) = 1, t \leq 0 \) and \( f_1(t) = 0, t \geq 1 \);
(ii) \( f_1 \in C^\infty(\mathbb{R}) \);
(iii) \( f_1^2 + f_2^2 = 1 \);
(iv) \( |f_1'(t)|, |f_2'(t)| \leq 2 \).

Define cutoff functions \( \chi_1, \chi_2 : \mathbb{R}^3 \to [0, 1] \) as \( \chi_1(x) = f_1(|x| - r) \) and \( \chi_2(x) = f_2(|x| - r) \). Let \( \varphi_1 = \chi_1 \varphi \) and \( \varphi_2 = \chi_2 \varphi \). Use a similar argument leading to (3.1), we have

\[
E(\varphi) \leq E(\varphi_1) + E(\varphi_2).
\]

Let us compare the three terms in the definition of the energy (1.1). We start with

\[
|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2 - |\nabla \varphi|^2 = |\nabla (\chi_1 \varphi)|^2 + |\nabla (\chi_2 \varphi)|^2 - |\nabla \varphi|^2
= \varphi^2 (|\nabla \chi_1|^2 + |\nabla \chi_2|^2) + 2 (\chi_1 \partial_r \chi_1 \varphi \partial_r \varphi + \chi_2 \partial_r \chi_2 \varphi \partial_r \varphi),
\]

where \( \partial_r \) is the derivative in the radial direction. Note that

\[
\chi_1 \partial_r \chi_1 + \chi_2 \partial_r \chi_2 = \frac{1}{2} \partial_r (\chi_1^2 + \chi_2^2) = 0.
\]

Therefore,

\[
|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2 - |\nabla \varphi|^2 = \varphi^2 (|\nabla \chi_1|^2 + |\nabla \chi_2|^2) \lesssim \varphi^2 1_{B_{r+1} \setminus B_r}.
\]

Integrating the last inequality, we obtain

\[
\int_{\mathbb{R}^3} |\nabla \varphi_1|^2 + |\nabla \varphi_2|^2 - |\nabla \varphi|^2 \lesssim \int_{B_{r+1} \setminus B_r} \varphi^2.
\]

For the nonlinear term, using Lemma 6, we estimate

\[
F(\varphi_1^2) + F(\varphi_2^2) - F(\varphi^2) \leq -F(\varphi^2) 1_{B_{r+1} \setminus B_r} \lesssim \varphi^2 1_{B_{r+1} \setminus B_r}.
\]

Therefore,

\[
\int_{\mathbb{R}^3} F(\varphi_1^2) + F(\varphi_2^2) - F(\varphi^2) \lesssim \int_{B_{r+1} \setminus B_r} \varphi^2.
\]

For the Coulomb interaction,

\[
D(\varphi_1^2, \varphi_1^2) + D(\varphi_2^2, \varphi_2^2) - D(\varphi^2, \varphi^2) = -2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\varphi_1^2(x) \varphi_2^2(y)}{|x-y|} \, dx 
\]

\[
\leq -2 \int_{B_r \setminus B_{r+1}} \frac{\varphi_1^2(x) \varphi_2^2(y)}{|x-y|} \, dx 
\]

\[
= -2 \int_{B_r \setminus B_{r+1}} \frac{\varphi^2(x) \varphi^2(y)}{|x-y|} \, dx 
\]

\[
\leq -\frac{2}{r+R} \int_{B_r \setminus B_{r+1}} \varphi^2(x) \varphi^2(y) \, dx 
\]

where we have used the fact that for \( x \in B_r \) and \( y \in B_{r+1} \setminus B_{r+1} \), \( |x-y| \leq r+R \).

The lemma follows by combining (3.3), (3.4), (3.5), and (3.6). \( \square \)
Lemma 8. There exists a universal constant $C_0$, such that for any ball $B_{R_0}(x)$ with $R_0 > 1$ and 
\[ \int_{B_{R_0}(x)} \phi^2 \geq C_0, \]
it holds 
\[ (3.7) \quad m = \int_{\mathbb{R}^3} \phi^2 \leq 2 \int_{B_{2R_0}(x)} \phi^2. \]

Proof. Without loss of generality, we assume $x = 0$ to simplify notations. Consider balls $B_r$ concentric with $B_{R_0}$. Denote for $r \geq 0$
\[ V(r) = \int_{B_r} \phi^2. \]

From Lemma 7, we have for any $R > 1$ and $r \in (0, R - 1)$,
\[ V(r + 1) - V(r) \geq \frac{V(r)(V(R) - V(r + 1))}{R + r}. \]
Hence, for $R > 4$ and $r \in [R/4, R/2 - 1]$ (so that $r < R - 1$), by monotonicity of $V(r)$,
\[ (3.9) \quad V(r + 1) - V(r) \geq \frac{V(r)(V(R) - V(r + 1))}{R} \geq \frac{V(R/4)(V(R) - V(R/2))}{R}. \]

Integrating $r$ from $R/4$ to $R/2 - 1$, we obtain
\[ (3.10) \quad V(R/2) - V(R/4) \geq \frac{V(R/4)}{C_1} (V(R) - V(R/2)), \]
for some universal constant $C_1$. The inequality holds for any $R > 4$, in particular, for any $k \geq 0$, we have
\[ V(2^{k-1}R) - V(2^{k-2}R) \geq \frac{V(2^{k-2}R)}{C_1} (V(2^kR) - V(2^{k-1}R)) \geq \frac{V(R/4)}{C_1} (V(2^kR) - V(2^{k-1}R)). \]
Choose $C_0 = 2C_1$. Take $R = 4R_0 > 4$, we then have for any $k \geq 0$,
\[ V(2^kR) - V(R/2) = \sum_{i=0}^{k} (V(2^iR) - V(2^{i-1}R)) \]
\[ \leq \sum_{i=0}^{k} \left( \frac{C_1}{V(R/4)} \right)^{i+1} (V(R/2) - V(R/4)) \leq V(R/2) - V(R/4), \]
where in the last inequality, we have used $V(R/4) \geq V(R_0) \geq C_0 = 2C_1$. Therefore
\[ (3.13) \quad 2V(2R_0) = 2V(R/2) \geq V(2^kR) = \int_{B_{2^kR}} \phi^2 \]
for any $k$. The lemma is proved by taking $k \to \infty$. \qed
Proof of Theorem 1. Assume the minimizer $\varphi$ exists with $\int \varphi^2 = m$. Let
\begin{equation}
R_0 = \inf_R \left\{ \exists x \in \mathbb{R}^3 \text{ s.t. } \int_{B_R(x)} \varphi^2 \geq C_0 \right\},
\end{equation}
where $C_0$ is the constant in Lemma 8. It is clear that $R_0 \in (0, \infty)$ for $m > C_0$. By Lemma 8, we have
\begin{equation}
m \leq 2 \int_{B_{2R_0}(x)} \varphi^2.
\end{equation}
Consider the ball $B_{2R_0}(x)$, there exists a universal number $n$ (in particular, independent of $R_0$) such that we can find $x_1, x_1, \cdots, x_n \in B_{2R_0}(x)$ with
\begin{equation}
B_{2R_0}(x) \subset \bigcup_{i=1}^n B_{R_0/2}(x_i).
\end{equation}
Hence,
\begin{equation}
\int_{B_{2R_0}(x)} \varphi^2 \leq \sum_{i=1}^n \int_{B_{R_0/2}(x_i)} \varphi^2 \leq n C_0.
\end{equation}
Combined with (3.15), we get $m \leq n C_0$. \hfill $\square$

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References


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