ISOLATING RANKIN-SELBERG LIFTS

JAYCE R. GETZ AND JAMIE KLASSEN

Abstract. Let $F$ be a number field and let $\pi$ be a cuspidal unitary automorphic representation of $\text{GL}_{mn}(\mathbb{A}_F)$ where $m$ and $n$ are integers greater than one. We propose a conjecturally necessary condition for $\pi$ to be a Rankin-Selberg transfer of an automorphic representation of $\text{GL}_m \times \text{GL}_n(\mathbb{A}_F)$. As evidence for the conjecture we prove the corresponding statement about automorphic $L$-parameters.

1. Introduction

Let $F$ be a number field and $\mathbb{A}_F$ its ring of adeles. A deep result in the theory of automorphic representations is the statement that an irreducible self-dual unitary cuspidal automorphic representation $\pi$ of $\text{GL}_{2n}(\mathbb{A}_F)$ with trivial central character is an endoscopic transfer of a cuspidal automorphic representation of $\text{SO}_{2n+1}(\mathbb{A}_F)$ if and only if the exterior square $L$-function $L(s, \pi, \wedge^2)$ has a pole at $s = 1$ (which is necessarily simple) [CKPSS] [GRS] [A]. There is a similar result for symplectic groups involving the symmetric square $L$-function $L(s, \pi, \text{Sym}^2)$.

It is natural to ask if the image of other functorial transfers of automorphic representations can be characterized in terms of $L$-functions. Apart from its intrinsic interest, such a characterization (even if it is conjectural) might be used to establish new cases of functoriality using Langlands’ idea of beyond endoscopy [La]. The reader is referred to [FLN], [FN], [G] for more details and to [V] [H] for examples where functorial transfers (known previously by different methods) are obtained using the idea of beyond endoscopy.

In this paper we restrict ourselves to the following (still conjectural) case of Langlands functoriality. Denote by

\begin{equation}
RS := RS_{m,n} : \text{GL}_m/\mathbb{C} \times \text{GL}_n/\mathbb{C} \longrightarrow \text{GL}_{mn}/\mathbb{C}
\end{equation}

the tensor product representation. Recall that the $L$-group of $\text{GL}_{n/F}$ is $^L\text{GL}_n := W'_F \times \text{GL}_n(\mathbb{C})$, where $W'_F$ is the Weil-Deligne group of $F$. Thus the neutral component of $^L\text{GL}_n$ is $\text{GL}_n(\mathbb{C})$, and we have an $L$-map

\[ RS := RS_{m,n} : ^L(\text{GL}_m \times \text{GL}_n) \longrightarrow ^L\text{GL}_{mn} \]

given by the identity map on the Weil-Deligne group factors and the tensor product on the neutral components. The (conjectural) functorial transfer attached to this map of $L$-groups

---

2010 Mathematics Subject Classification. Primary 11F66, Secondary 20G05.
is the Rankin-Selberg transfer from $\GL_m \times \GL_n(\mathbb{A}_F)$ to $\GL_{mn}(\mathbb{A}_F)$, so named because of the identity

$$L(s, \pi_1 \times \pi_2, RS) = L(s, \pi_1 \times \pi_2) \quad (1.0.2)$$

where $\pi_1$ (resp. $\pi_2$) is an automorphic representation of $\GL_m(\mathbb{A}_F)$ (resp. $\GL_n(\mathbb{A}_F)$), the $L$-function on the left is the Langlands $L$-function and the $L$-function on the right is the Rankin-Selberg $L$-function. If $\pi$ is an isobaric automorphic representation of $\GL_{mn}(\mathbb{A}_F)$ we say that it is a Rankin-Selberg transfer (from $\GL_m \times \GL_n(\mathbb{A}_F)$) if it is a transfer of an automorphic representation of $\GL_m \times \GL_n(\mathbb{A}_F)$ with respect to $RS$.

**Remark.** We now make a remark (no doubt well-known to experts) on the importance of the Rankin-Selberg lift. Consider the set of all $L$-maps

$$L^G \rightarrow L^{\GL_n}$$

that are the identity map on the Weil-Deligne group factors. We claim that Langlands functoriality for this set of $L$-maps follows from Langlands functoriality for $RS$ for all $m$ and $n$ together with Langlands functoriality for the set of $L$-maps $L^G \rightarrow L^{\GL_n}$ trivial on the Weil-Deligne group factors where $L^G$ is simple and the $L$-map is induced by a fundamental representation of $L^G$. Indeed, one can first reduce to the case where $L^G$ is simple using the fact that a representation of an almost direct product of simple algebraic groups is a tensor product of representations of the factors. One then reduces to fundamental representations using the fact that any representation of a simple algebraic group is a subrepresentation of a tensor product of fundamental representations.

Our aim in this paper is to give some Galois-theoretic evidence for a conjecturally necessary condition for a cuspidal automorphic representation to be in the image of the (conjectural) functorial transfer induced by $RS$. To state it, let

$$r_{m,n} : \GL_{mn}/\mathbb{C} \rightarrow \GL(V_{m,n}) \quad (1.0.3)$$

be the representation of algebraic groups over $\mathbb{C}$ given by

$$V_{m,n} = (\mathbb{C}^{mn})^{(2^i1^{mn-i})}$$

Here and below $(\mathbb{C}^N)^\lambda$ is the irreducible representation of $\GL_N$ obtained by applying the Schur functor attached to the partition $\lambda$ of $N$. Moreover for positive integers $a_i, b_i$ with $1 \leq i \leq k$ and $a_i > a_{i+1}$ for all $1 \leq i \leq k - 1$ the notation $(a_1^{b_1}, \cdots, a_k^{b_k})$ is shorthand for the partition of $\sum_{i=1}^k b_i a_i$ given by

$$\underbrace{a_1 + \cdots + a_1}^{b_1 \text{ times}} + \underbrace{a_{k} + \cdots + a_k}_{b_k \text{ times}}.$$

We use the same symbol $r_{m,n}$ to denote the $L$-map

$$r_{m,n} : L^{\GL_{mn}} \rightarrow \GL(V_{m,n}(\mathbb{C})) \quad (1.0.4)$$
that is trivial on the Weil-Deligne group and given by \( r_{m,n} \) on the neutral factor.

Let \( (F^\times \backslash \mathbb{A}^\times_F)^\wedge \) denote the group of abelian characters of \( F^\times \backslash \mathbb{A}^\times_F \). Here and below we take the convention that abelian characters are unitary. We make the following conjecture:

Conjecture 1.1. Let \( m, n \in \mathbb{Z}_{>1} \). Let \( \pi = \pi_1 \boxplus \cdots \boxplus \pi_k \) be an isobaric automorphic representation of \( \text{GL}_{mn}(\mathbb{A}_F) \) where each \( \pi_i \) is unitary and cuspidal. If \( \pi \) is a Rankin-Selberg transfer from \( \text{GL}_m \times \text{GL}_n(\mathbb{A}_F) \) then

\[
L(s, \pi, r_{m,n} \otimes \chi)
\]

has a pole at \( s = 1 \) for some \( \chi \in (F^\times \backslash \mathbb{A}^\times_F)^\wedge \). If \( m = n \) the same assertion is true if we replace \( r_{m,n} \) by \( \text{Sym}^n \).

In the conjecture we are implicitly assuming that \( L(s, \pi, r_{m,n} \otimes \chi) \) (resp. \( L(s, \pi, \text{Sym}^n \otimes \chi) \)) has a meromorphic continuation to a neighborhood of \( s = 1 \) for all \( \chi \). Of course, this is predicted by Langlands functoriality but is far from being proven in general.

As evidence for the conjecture, we prove a Galois-theoretic analogue of it. Let

\[
\rho : W'_F \rightarrow {}^L\text{GL}_{mn}
\]

be an \( L \)-parameter. We say that \( \rho \) is a Rankin-Selberg transfer if there is an \( L \)-parameter \( \sigma : W'_F \rightarrow {}^L(\text{GL}_m \times \text{GL}_n) \) and a \( g \in {}^L\text{GL}_{mn} \) such that

\[
\rho(w) = gRS \circ \sigma(w)g^{-1}
\]

for all \( w \in W'_F \). Because we have allowed conjugation by \( g \in {}^L\text{GL}_{mn} \) in this definition the statement that \( \rho \) is a Rankin-Selberg transfer depends only on the equivalence class of the \( L \)-parameter \( \rho \).

Remark. Since the \( L \)-group of \( \text{GL}_d \) is the direct product of \( W'_F \) and \( \text{GL}_d(\mathbb{C}) \), the reader will lose nothing if he or she replaces \( {}^L\text{GL}_d \) by \( \text{GL}_d(\mathbb{C}) \) (resp. \( {}^L(\text{GL}_m \times \text{GL}_n) \) by \( \text{GL}_m \times \text{GL}_n(\mathbb{C}) \)) in the discussion above.

We prove the following theorem:

Theorem 1.2. Let \( m, n \in \mathbb{Z}_{>1} \). Let \( \rho : W'_F \rightarrow {}^L\text{GL}_{mn} \) be a semisimple \( L \)-parameter such that \( r_{m,n} \circ \rho \) is automorphic. If \( \rho \) is a Rankin-Selberg transfer then

\[
L(s, r_{m,n} \circ \rho \otimes \chi)
\]

has a pole at \( s = 1 \) for some \( \chi \in (F^\times \backslash \mathbb{A}^\times_F)^\wedge \). If \( m = n \) the same assertion is true if we instead assume that \( \text{Sym}^n \circ \rho \) is automorphic and replace \( r_{m,n} \) by \( \text{Sym}^n \).

Here when we say that an \( L \)-parameter \( \sigma \) is automorphic we mean that it is semisimple and if \( \sigma = \oplus_{i=1}^k \sigma_i \) is a decomposition of \( \sigma \) into irreducible subrepresentations, then there
is a set of unitary cuspidal automorphic representations $\pi_1, \ldots, \pi_k$ such that $\sigma_i$ is the $L$-parameter of $\pi_i$ for all $1 \leq i \leq k$. In other words we require that $\sigma_v$ is the $L$-parameter of $\pi_v$ for all places $v$ of $F$ and all $1 \leq i \leq k$. Thus if $\sigma$ is automorphic then $L(s, \sigma)$ has a meromorphic continuation to the entire complex plane that is holomorphic apart from finitely many poles on the line $\Re(s) = 1$.

Remark. The representation $r_{m,n}$ is not necessarily the representation of minimal degree such that the conclusion of the theorem holds. For example,

$$\dim \text{Sym}^2(C^4) < \dim r_{2,2}.$$  

The reason for isolating this particular family of representations is simply that they have the properties we require to prove Theorem 1.2 for all $m$ and $n$. They also satisfy assertion (2) of Theorem 3.6, which ought to be helpful for applications and refinements of our result.

We now outline the contents of this paper. In §2 we collect some results on Schur-Weyl duality, partially for the purpose of setting notation. In §3 we prove some representation theoretic results, and in §4 we prove Theorem 1.2 using these results. In this paper all algebraic groups are affine and all morphisms of algebraic groups are assumed to be morphisms in the category of algebraic groups over $\mathbb{C}$ (i.e. morphisms of affine group schemes of finite type over $\text{Spec}(\mathbb{C})$).

2. RECALLECTIONS ON SCHUR-WEYL DUALITY

In the following, we will use Schur-Weyl duality; we now recall the relevant notation. Throughout this paper $V$ and $W$ denote finite dimensional vector spaces over $\mathbb{C}$. Let $\mathfrak{S}_d$ be the symmetric group on $d$ letters. We write $\lambda \vdash d$ to indicate that $\lambda$ is a partition of $d$ and $|\lambda|$ will denote the number partitioned by $\lambda$, so that $\lambda \vdash |\lambda|$. Schur-Weyl duality is the statement that there is a decomposition into irreducibles

$$V^\otimes d = \bigoplus_{\lambda \vdash d} V^\lambda \otimes S_\lambda$$

as $\text{GL}_n \times \mathfrak{S}_d$-modules, where $V^\lambda$ denotes the Schur functor corresponding to $\lambda$ applied to $V$ and $S_\lambda$ denotes the Specht module corresponding to $\lambda$. As representations of $\text{GL}(V)$ (resp. $\mathfrak{S}_d$) $V^\lambda \cong V^\mu$ (resp. $S_\lambda \cong S_\mu$) if and only if $\lambda = \mu$.

With the notation above, one has the following definition:

**Definition 2.1.** For partitions $\lambda, \mu, \nu$ of $d$ the **Kronecker coefficient** $k_{\lambda\mu\nu}$ is the unique (nonnegative) integer such that

$$(2.0.1) \quad S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash d} S_\nu^{\oplus k_{\lambda\mu\nu}}.$$  

as $\mathfrak{S}_d$-representations.

By duality the Kronecker coefficients also determine a branching rule for $\text{GL}_n$:
Proposition 2.2. One has
\[(V \otimes W)^\lambda|_{RS(GL(V) \times GL(W))} = \bigoplus_{\mu, \nu \vdash |\lambda|} (V^\mu \otimes W^\nu)^{\otimes k_{\lambda \mu \nu}}\]
as representations of GL(V) x GL(W).

Proof. Let \(d > 0\). Schur-Weyl duality gives us the decomposition
\[(V \otimes W)^{\otimes d} = \bigoplus_{\lambda \vdash d} (V \otimes W)^{\lambda} \otimes S_{\lambda}\]
as GL(V \otimes W) x S_d-modules. On the other hand, there is a GL(V) x GL(W) x S_d-equivariant isomorphism \((V \otimes W)^{\otimes d} \cong V^{\otimes d} \otimes W^{\otimes d}\) and we can apply Schur-Weyl duality twice again to obtain
\[(V \otimes W)^{\otimes d} \cong V^{\otimes d} \otimes W^{\otimes d} \cong \left( \bigoplus_{\mu \vdash d} V^\mu \right) \otimes \left( \bigoplus_{\nu \vdash d} W^\nu \right) \cong \bigoplus_{\mu, \nu \vdash d} V^\mu \otimes W^\nu \otimes (S_{\mu} \otimes S_{\nu})\]
\[\cong \bigoplus_{\mu, \nu \vdash d} V^\mu \otimes W^\nu \otimes \left( \bigoplus_{\lambda \vdash d} S_{\lambda}^{\otimes k_{\lambda \mu \nu}} \right)\]
\[\cong \bigoplus_{\lambda \vdash d} \left( \bigoplus_{\mu, \nu \vdash d} (V^\mu \otimes W^\nu)^{\otimes k_{\lambda \mu \nu}} \right) \otimes S_{\lambda}\]
as GL(V) x GL(W) x S_d-modules. Comparing terms in the two different decompositions of \((V \otimes W)^{\otimes d}\) one has
\[(V \otimes W)^\lambda|_{RS(GL(V) \times GL(W))} = \bigoplus_{\mu, \nu \vdash |\lambda|} (V^\mu \otimes W^\nu)^{\otimes k_{\lambda \mu \nu}}\]
as required. \(\square\)

Another helpful property of the Kronecker coefficients is their symmetry with respect to the three arguments. To prove a precise statement along these lines we set some notation. For all partitions \(\lambda \vdash d\) let
\[\chi_{\lambda} : \mathbb{C}[S_d] \longrightarrow \mathbb{C}\]
be the character of \(S_{\lambda}\) (defined on the group algebra \(\mathbb{C}[S_d]\) of \(S_d\)) and let
\[\langle \cdot, \cdot \rangle : \mathbb{C}[S_d] \times \mathbb{C}[S_d] \longrightarrow \mathbb{C}\]
\[\langle f, g \rangle \longmapsto \frac{1}{d!} \sum_{\sigma \in S_d} f(\sigma)\overline{g(\sigma)}\]
be the usual invariant inner product. Finally if \(\lambda \vdash d\) denote by \(C_{\lambda}\) the corresponding conjugacy class in \(S_d\); it consists of those permutations whose cycle type is given by \(\lambda\).

One has the following proposition:
Proposition 2.3. If \( \lambda, \mu, \nu \vdash d \) then
\[
k_{\lambda \mu \nu} = \frac{1}{d!} \sum_{\eta \vdash d} |C_{\eta}| \chi_\lambda(C_{\eta}) \chi_\mu(C_{\eta}) \chi_\nu(C_{\eta}).
\]

Proof. By (2.0.1) we have
\[
\langle \chi_\lambda \chi_\mu, \chi_\nu \rangle = \sum_{\eta \vdash d} k_{\lambda \mu \eta} \chi_\eta = k_{\lambda \mu \nu}.
\]
On the other hand it is well-known that the characters of \( S_d \) are real-valued. Thus
\[
\langle \chi_\lambda \chi_\mu, \chi_\nu \rangle = 1 = \sum_{\eta \vdash d} k_{\lambda \mu \eta} \chi_\eta = \frac{1}{d!} \sum_{\sigma \in S_d} \chi_\lambda(\sigma) \chi_\mu(\sigma) \chi_\nu(\sigma).
\]
This gives the result. \( \square \)

Recall that for any partition \( \lambda = \lambda_1 \geq \cdots \geq \lambda_k \) we can form its conjugate \( \lambda' \) where
\[
\lambda'_i = |\{ \lambda_k : \lambda_k \geq i \}|.
\]
Yet another helpful fact about the Kronecker coefficients is their conjugate symmetry:

Proposition 2.4. One has \( k_{\lambda \mu \nu} = k_{\lambda' \mu' \nu'} \).

Proof. Recall that \( \chi_{(1^d)} : S_d \to \mathbb{C} \) is the sign homomorphism. It is known [FH, §4.1, Exercise 4.4] that if \( \lambda \vdash d \) then \( \chi_{\lambda'} = \chi_{\lambda(1^d)} \). Thus using the fact that the characters of \( S_d \) are real valued we have we have
\[
k_{\lambda \mu \nu} = \frac{1}{d!} \sum_{\eta} |C_{\eta}| \chi_\lambda(C_{\eta}) \chi_\mu(C_{\eta}) \chi_\nu(C_{\eta}) = \frac{1}{d!} \sum_{\eta} |C_{\eta}| (\chi_{(1^d)}(C_{\eta}))^2 \chi_\lambda(C_{\eta}) \chi_\mu(C_{\eta}) \chi_\nu(C_{\eta})
\]
\[
= \langle \chi_\lambda \chi_\mu \chi_{(1^d)} \chi_\nu \chi_{(1^d)} \rangle
\]
\[
= \langle \chi_\lambda \chi_\mu \chi_{\nu} \rangle = k_{\lambda' \mu' \nu'}.
\]

\( \square \)

3. Main representation-theoretic results

In this section we prove the following result

Theorem 3.1. Let \( m, n \in \mathbb{Z}_{\geq 1} \). The restriction of the representation \( r_{m,n} : GL_{mn} \to GL(V_{m,n}) \) of (1.0.3) to \( RS(GL_m \times GL_n) \) stabilizes a unique one-dimensional subspace. If \( m = n \) the restriction of the representation \( Sym^n : GL_{n^2} \to GL(Sym^n(\mathbb{C}^n)) \) to \( RS(GL_m \times GL_n) \) stabilizes a unique one dimensional subspace.

Theorem 3.1 is an immediate consequence of Theorem 3.4, Corollary 3.5 and Theorem 3.6 below. These latter three results are more precise in that they realize \( RS(GL_m \times GL_n) \) as the neutral component of the stabilizer of a specific vector in \( V_{m,n} \).

To prove these refined statements, it is useful to first prove the following lemma:
Lemma 3.2. Let $K$ be a connected algebraic group such that $RS(GL_m \times GL_n) \leq K \leq GL_{mn}(\mathbb{C})$. Then $K = RS(GL_m \times GL_n)$ or $K = GL_{mn}$. In other words, $RS(GL_m \times GL_n)$ is maximal among the connected algebraic subgroups of $GL_{mn}$.

Proof. To ease notation write $H := RS(GL_m \times GL_n)$ for the proof. By the general theory of algebraic groups, since $K$ is connected the derived subgroup $[K, K]$ of $K$ is again connected. Furthermore we have $[H, H] \leq [K, K] \leq [GL_{mn}, GL_{mn}]$. By [Dyn, Theorem 1.3] the group $[H, H]$ is isomorphic to $SL_m \times SL_n$ is maximal among the connected subgroups of $[GL_{mn}, GL_{mn}] = SL_{mn}$. Thus we must have $[K, K] = SL_{mn}$ or $[K, K] = SL_m \times SL_n$. Moreover, in either case since the subgroup $\mathbb{G}_m I_{mn}$ of scalar matrices is contained in $H$ it must be contained in $K$. If $[K, K] = SL_{mn}$ this implies that $K = GL_{mn}$ and we are done in this case.

Suppose $[K, K] = SL_m \times SL_n$. Denote by $R(G)$ the radical of an algebraic group $G$. Then $R(K) = R(K) \cap Z(GL_{mn}).R(K) \cap SL_{mn} = \mathbb{G}_m I_{mn}.R(K) \cap SL_{mn}$

Since $[K, K]$ is maximal among connected subgroups of $SL_{mn}$, the neutral component $(R(K) \cap SL_{mn})^\circ$ of $R(K) \cap SL_{mn}$ is trivial and we conclude that $R(K) = \mathbb{G}_m I_{mn}$, the diagonal matrices. Hence $K = H$ as asserted.

Here it is also useful to classify the one-dimensional representations of $GL_n$ in terms of Schur functors; this change in language will make clear the applicability of the above results on Schur-Weyl duality.

Lemma 3.3. Let $V \cong \mathbb{C}^n$ and let $\rho : GL(V) \rightarrow GL(W)$ be a one-dimensional representation. Then $\rho(g) = (\det g)^k$ for some $k \in \mathbb{Z}$. Moreover if $W$ is the image of $V$ under a Schur functor and $\rho$ the action induced from the natural action of $GL_n$ on $V$, then $k$ is nonnegative and $W = (\wedge^n V)^{ \otimes k} = V^{(k^n)}$.

Proof. Since $W$ is one-dimensional we can identify $GL(W) \cong \mathbb{G}_m$. Then $\rho$ is a morphism $GL_n \rightarrow \mathbb{G}_m$. Since $\mathbb{G}_m$ is abelian, $\rho$ factors through the derived subgroup $[GL_n, GL_n] = SL_n$. It follows that $\rho = \phi \circ \det$ for some $\phi \in \text{End}(\mathbb{G}_m) \cong \mathbb{Z}$, hence $\rho = \det^k$ for some $k \in \mathbb{Z}$.

According to [Ful, Theorem 2, §8.2], applying a Schur functor to the standard representation of $GL_n$ yields a polynomial representation. Thus if $W$ is the image of a Schur functor and $\rho$ the induced action, then $k \geq 0$ and $W \cong (\wedge^n V)^{ \otimes k}$. If we use the “categorical” definition of Schur functors given in [Ful, beginning of §8.1], it is easy to see that both $(\wedge^n V)^{ \otimes k}$ and $V^{(k^n)}$ satisfy the same universal property and therefore must be isomorphic. \qed

We have the following theorem:
Theorem 3.4. Let $V \cong \mathbb{C}^n$. The only irreducible polynomial representation of $GL(V \otimes V)$ whose restriction to $RS(GL_n \times GL_n)$ contains a subspace isomorphic to $\wedge^n V \otimes \wedge^n V$ is $\text{Sym}^n$. This subspace occurs with multiplicity one.

Proof. Recall that $V^{(1^*n)} = \wedge^n V$ and $V^{(n)} = \text{Sym}^n V$, and that irreducible polynomial representations of $GL(V \otimes V)$ are all obtained by applying Schur functors to $V \otimes V$ [Ful, Theorem 2, §8.2]. Thus by Proposition 2.2 it suffices to show that $k_{(n)(1^*n)(1^*n)} = 1$ and $(n)$ is the only partition of $n$ such that $k_{\lambda(1^*n)(1^*n)} \geq 1$.

Recall that $\chi_{(1^*n)}(\sigma) = \text{sgn}(\sigma)$ and hence for all $\sigma \in S_n$ one has $\chi_{(1^*n)}(\sigma)\chi_{(1^*n)}(\sigma) = 1 = \chi_{(n)}(\sigma)$ (the trivial character). Thus by Proposition 2.3 one has

$$k_{\lambda(1^*n)(1^*n)} = \frac{1}{n!} \sum_{\eta \vdash n} |C_\eta| \chi_\lambda(C_\eta) \chi_{(1^*n)}(C_\eta) \chi_{(1^*n)}(C_\eta) = \frac{1}{n!} \sum_{\eta \vdash n} |C_\eta| \chi_\lambda(C_\eta) \chi_{(n)}(C_\eta) = \langle \chi_\lambda, \chi_{(n)} \rangle.$$ 

It follows that if $k_{\lambda(1^*n)(1^*n)} \geq 1$ then $\lambda = (n)$, and moreover that $k_{(n)(1^*n)(1^*n)} = 1$. 

If $V$ is the standard representation of $GL_n$ then

$$\text{Sym}^n(V \otimes V)|_{RS(GL_n \times GL_n)} = \bigoplus_{\lambda \vdash n} V^\lambda \otimes V^\lambda$$

[GW, Corollary 5.6.8]. We can fix an isomorphism

$$V^\vee \otimes V^\vee \longrightarrow M_n$$

(where $M_n$ is the space of square $n \times n$ matrices) so that $\text{Sym}^n(V \otimes V)$ is intertwined with the space of symmetric polynomials $\mathcal{P}^n(M_n)$ on $M_n$ of degree $n$ with $GL_n \times GL_n$ action given by

$$(y, z)p(x) = p(y^txz)$$

(compare the proof of [GW, Corollary 5.6.8]). By Lemma 3.3, the only partition of $n$ which furnishes a Schur functor giving a one-dimensional representation of $GL_n$ when applied to $V$ is $(1^*n)$. Since all of the subrepresentations in the decomposition of (3.0.1) correspond to partitions of $n$, the only one-dimensional subrepresentation is $\wedge^n V \otimes \wedge^n V$. Thus we can conclude that under the identification (3.0.2) the subspace $\wedge^n V \otimes \wedge^n V$ is sent to the line spanned by $\text{det}$.

For the moment, identify $\text{Sym}^n(V \otimes V) = \mathcal{P}_n(M_n)$. Suppose $g \in GL(V \otimes V)$ has the function $\text{det}$ as an eigenvector in the representation $\text{Sym}^n(V \otimes V) = \mathcal{P}_n(M_n)$; equivalently, there exists nonzero $\lambda \in \mathbb{C}$ such that $\text{det}(g \cdot x) = \lambda \text{det}(x)$ for all $x \in M_n$. In particular, if $\text{det}(x) \neq 0$ then $\text{det}(g \cdot x) \neq 0$, so $g$ preserves the set of invertible matrices. Then by [MP, Theorem 2.1], there exist $y, z \in GL_n$ such that

$$g \cdot x = y^txz$$

for all $x \in M_n$ or else

$$g \cdot x = y^tx^tz.$$
for all \( x \in M_n \); i.e. the full stabilizer of \( \det \) in this representation is given by the product

\[
\{ \text{Id}, x \mapsto x^t \} RS(\text{GL}_n \times \text{GL}_n).
\]

This implies the following refinement of Theorem 3.4:

**Corollary 3.5.** Let \( r : G \to \text{GL}_{n^2} \) a representation. The group \( r(G) \) stabilizes \( \det \) in the representation \( \text{Sym}^n(V \otimes V) \cong \mathcal{P}^n(M_n) \) if and only if

\[
r(G) \leq \{ \text{Id}, x \mapsto x^t \} RS(\text{GL}_n \times \text{GL}_n).
\]

□

Applying [GW, Corollary 5.6.8] again one has

\[
\text{Sym}^k((\mathbb{C}^m \otimes \mathbb{C}^n)|_{RS(\text{GL}_m \times \text{GL}_n)}) = \bigoplus_{\lambda \vdash k} (\mathbb{C}^m)^{\lambda} \otimes (\mathbb{C}^n)^{\lambda}.
\]

If one of the summands \((\mathbb{C}^m)^{\lambda} \otimes (\mathbb{C}^n)^{\lambda}\) on the right-hand side were one-dimensional, necessarily both \((\mathbb{C}^m)^{\lambda}\) and \((\mathbb{C}^n)^{\lambda}\) must be one-dimensional. By Lemma 3.3 we know that \((\mathbb{C}^m)^{\lambda}\) is one-dimensional if and only if \(\lambda = (s_1^m)\) for some positive integer \(s_1\) and \((\mathbb{C}^n)^{\lambda}\) is one-dimensional if and only if \(\lambda = (s_2^n)\) for some positive integer \(s_2\). If \(m \neq n\) then these conditions are clearly incompatible.

Thus in this case the symmetric powers \(\text{Sym}^k((\mathbb{C}^m \otimes \mathbb{C}^n)\) do not furnish us with lines invariant under the subgroup \(RS(\text{GL}_m \times \text{GL}_n)\); we must investigate other representations of \(\text{GL}_{mn}\). As in the introduction, let \((r_{m,n}, V_{m,n})\) be the representation of \(\text{GL}_{mn}\) defined using the Schur functor attached to the partition \((2^2,1^{mn-4})\). Using work of Ballantine and Orellana [BO] we prove the following theorem:

**Theorem 3.6.** Let \(m, n \in \mathbb{Z}_{\geq 1}\).

1. One has \(k_{(m^n)(m^n)}(2^2,1^{mn-4}) = 1\).
2. The subspace of \(V_{m,n}\) corresponding to \((\mathbb{C}^m)^{(m^n)} \otimes (\mathbb{C}^n)^{(m^n)}\) under the decomposition of \(V_{m,n}\) given by Proposition 2.2 is the only one-dimensional subrepresentation of \(r_{m,n}|_{RS(\text{GL}_m \times \text{GL}_n)}\).
3. Suppose \(r : G \to \text{GL}_{mn}\) is a representation of a connected algebraic group. Then \(r(G)\) stabilizes this one-dimensional subspace if and only if \(r(G) \subset RS(\text{GL}_m \times \text{GL}_n)\).

**Proof.**

1. By [BO, Corollary 4.6] one has \(k_{(mn-2,2)(m^n)(n^m)} = 1\). We have \(k_{(mn-2,2)(m^n)(n^m)} = k_{(m^n)(m^n)(mn-2,2)}\) by Proposition 2.3 and \(k_{(m^n)(m^n)(mn-2,2)} = k_{(m^n)(m^n)(2^2,1^{mn-4})}\) by Proposition 2.4.
2. By Proposition 2.2, the only subrepresentations of \(r_{m,n}|_{RS(\text{GL}_m \times \text{GL}_n)}\) are of the form

\[
(\mathbb{C}^m)^{\lambda} \otimes (\mathbb{C}^n)^{\mu}
\]
where $\lambda, \mu \vdash mn$. Such a subrepresentation is only one-dimensional if both $(\mathbb{C}^m)^\lambda$ and $(\mathbb{C}^n)^\mu$ are one-dimensional. Thus by Lemma 3.3, we have $\lambda = (k_1^m)$ and $\mu = (k_2^n)$ for some $k_1, k_2 \geq 0$. However since $\lambda, \mu \vdash mn$ the only possibility is $\lambda = (n^m), \mu = (m^n)$.

(3) This follows from Lemma 3.2.

□

4. Proof of Theorem 1.2

In this section we explain how to deduce Theorem 1.2, which we restate for the convenience of the reader:

**Theorem 4.1.** Let $m, n \in \mathbb{Z}_{\geq 1}$. Let $\rho : W'_F \rightarrow L^{\text{GL}_{mn}}$ be a semisimple $L$-parameter such that $r_{m,n} \circ \rho$ is automorphic. If $\rho$ is a Rankin-Selberg transfer then

$$L(s, r_{m,n} \circ \rho \otimes \chi)$$

has a pole at $s = 1$ for some $\chi \in (F^\times \backslash \mathbb{A}_F^\times)^\wedge$. If $m = n$ the same assertion is true if we assume instead that $\text{Sym}^n \circ \rho$ is automorphic and replace $r_{m,n}$ by $\text{Sym}^n$.

**Proof.** Consider the algebraic envelope $H$ of $\rho$. Thus $H$ is defined to be the Zariski closure of

$$\rho(W'_F) \cap L^{\text{GL}_{mn}} = \rho(W'_F) \cap \text{GL}_{mn}(\mathbb{C})$$

in $\text{GL}_{mn}(\mathbb{C})$. The group $H$ is reductive because $\rho$ is assumed to be semisimple; to see this one reduces to the case where $\rho$ is irreducible and then applies [BorT, Corollaire 3.9]. Recall that we assumed that $r_{m,n} \circ \rho$ is automorphic. Let

$$r_{m,n} \circ \rho = \bigoplus_{i=1}^n \sigma_i$$

be a decomposition of $\rho$ into irreducible subrepresentations and for each $i$ let $\pi_i$ be a cuspidal unitary automorphic representation with Langlands parameter $\sigma_i$. For $\chi \in (F^\times \backslash \mathbb{A}_F^\times)^\wedge$ we have

$$L(s, r_{m,n} \circ \rho \otimes \chi) = \prod_{i=1}^k L(s, \sigma_i \otimes \chi)$$

$$= \prod_{i=1}^k L(s, \pi_i \otimes \chi).$$

The latter product of $L$-functions has a pole at $s = 1$ for some $\chi \in (F^\times \backslash \mathbb{A}_F^\times)^\wedge$ if and only if $\pi_i$ is a Hecke-character for some $i$. This occurs if and only if $\sigma_i$ is one-dimensional for some $i$, which occurs if and only if $r_{m,n}(H)$ stabilizes a one-dimensional subspace of $V_{m,n}$.

Note that $\rho$ is a Rankin-Selberg transfer if and only if $H \subseteq g^{-1} \text{RS}(\text{GL}_m \times \text{GL}_n)g$ for some $g \in \text{GL}_{mn}(\mathbb{C})$. Thus, in particular, if $\rho$ is a Rankin-Selberg transfer Theorem 3.1 implies that $r_{m,n}(H)$ stabilizes a one-dimensional subspace of $V_{m,n}$. This completes the proof for the representations $r_{m,n}$. The proof with $r_{m,n}$ replaced by $\text{Sym}^n$ is exactly the same.
We close the paper with a remark about the \( \chi \) appearing in Theorem 4.1. Assume that \( \rho \) satisfies the hypotheses of Theorem 4.1; thus
\[
\rho \simeq RS(\rho_1 \times \rho_2)
\]
for a parameter \( \rho_1 \times \rho_2 : LW'_F \to L(\text{GL}_{m} \times \text{GL}_{n}) \). Then \( \rho \) acts on the one-dimensional subrepresentation of \( r_{m,n} \circ \rho \) given by Theorem 3.6 via \( \det(\rho_1)^n \otimes \det(\rho_2)^m \), so we can take
\[
\chi = \det(\rho_1)^{-n} \otimes \det(\rho_2)^{-m}
\]
in the statement of Theorem 4.1. Similarly, if if \( m = n \) and we replace \( r_{m,n} \) by \( \text{Sym}^n \) then we can take
\[
\chi = \det(\rho_1)^{-1} \otimes \det(\rho_2)^{-1} = \det(\rho)^{-1}
\]
in Theorem 4.1.

5. ACKNOWLEDGEMENTS

The authors thank the referee for encouraging them to explicate several statements and proofs.

REFERENCES

[Dyn] E. B. Dynkin, Maximal Subgroups of the Classical Groups, AMS Translations (2) 6 (1957) 245-378. 7
[GRS] D. Ginzburg, S. Rallis and D. Soudry, Generic automorphic forms on \( \text{SO}(2n+1) \): Functorial lift to \( \text{GL}(2n) \), endoscopy and base change, IMRN 14 (2001) 729-764. 1
[H] P. E. Herman *Quadratic Base Change and the Analytic Continuation of the Asai L-function: A new Trace formula approach*, submitted.


Department of Mathematics, Duke University, Durham, NC 27708-0320

E-mail address: jgetz@math.duke.edu

Department of Mathematics and Statistics, McGill University, Montreal, Quebec, Canada H3A 0B9

E-mail address: michigan.j.frog@gmail.com