The Four Subspaces

Recall: To any matrix $A$, we can associate:

- $\text{Col}(A)$: basis = pivot columns of $A$; $\text{dim}=\text{rank}$
- $\text{Nul}(A)$: basis = vectors in the PVF of $Ax=0$;
  $\text{dim}=\#\text{free vars}=\#\text{cols}-\text{rank}$

There are two more subspaces: just replace $A$ by $A^T$, then take $\text{Col}$ & $\text{Nul}$.

Why? Orthogonality $\Rightarrow$ least $\mathbb{D}$s (bear with me...)

Def: The row space of $A$ is $\text{Row}(A) = \text{Col}(A^T)$.

This is the subspace spanned by the rows of $A$, regarded as (row) vectors in $\mathbb{R}^n$.

This is a subspace of $\mathbb{R}^n$, $n=\#\text{columns}$
($n=\#\text{entries in each row}$)

$\Rightarrow$ row picture

Eg: $\text{Row}\begin{pmatrix}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix}1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix}4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix}7 \\ 8 \\ 9 \end{pmatrix}\right\}$

$= \text{Col}\begin{pmatrix}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Fact: Row operations do not change the row space.
Why? If the rows are $v_1, v_2, v_3$ then \( \text{Row}(A) = \text{Span}\{v_1, v_2, v_3\} \). Row ops:

- \( R_3 \leftarrow R_3 \): \( \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_3, v_2, v_3\} \)
- \( R_2 \times 3 \): \( \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, 3v_2, v_3\} \)
- \( R_2 \leftarrow 2R_1 \): \( \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2, v_3\} \)

because \( v_2 + 2v_1 \in \text{Span}\{v_1, v_2, v_3\} \)

and \( v_2 = (v_2 + 2v_1) - 2v_1 \in \text{Span}\{v_1, v_2, v_3\} \)

This is a col space (of \( A^T \)), so you know how to compute a basis (pivot columns of \( A^T \)). But you can also find a basis by doing elimination on \( A^T \):

Thm: The nonzero rows of any \( \text{REF} \) of \( A \) form a basis for \( \text{Row}(A) \).

Eg: \[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1 \\
1 & 2 & -1 & -2
\end{bmatrix}
\xrightarrow{\text{REF}}
\begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & 0 & -3 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Basis: \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \)
or: \[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1 \\
1 & 2 & -1 & -2
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(another) Basis: \[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

Proof:
1. Spans: row ops don't change Row\(A\), and you can always delete the zero vector without changing the span.

2. LI: \[0 = x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_1 \\ -3x_2 \end{pmatrix}\]

Solve by forward-substitution:
- \(=\) pivot, so this entry in the sum is just \(1\).
- \(=\) pivot, so this entry in the sum is just \((-3)\).

Consequence: \(\dim \text{Row}(A) = \#\text{pivot rows} = \#\text{pivots} = \text{rank}\).

(a non-zero row of an REF matrix has a pivot)
Def: The left null space of \( A \) is \( \text{Null}(A^T) \).

This is the solution set of \( A^T x = 0 \).

Notation: Just \( \text{Null}(A^T) \) (no new notation)

This is a subspace of \( \mathbb{R}^m \) \( m = \# \text{rows} \)
\( (m = \# \text{columns of } A^T) \)
\( \Rightarrow \) column picture

NB: \( A^T x = 0 \Leftrightarrow 0 = (A^T x)^T = x^T A \)

so \( \text{Null}(A^T) = \{ \text{row vectors } x \in \mathbb{R}^m : x^T A = 0 \} \)

\( \text{Null}(A^T) \) is a null space, so you know how to compute a basis (PFE of \( A^T x = 0 \)). You can also find a basis by doing elimination on \( A \):

Thm/Procedure: To compute a basis of \( \text{Null}(A^T) \):

1. Form the augmented matrix \( [A \mid \text{Im}] \)
2. Eliminate to REF
3. The rows on the right side of the line next to zero rows on the left form a basis of \( \text{Null}(A^T) \).
Example:

\[ A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftarrow 2R_2} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & -2 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_2} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

Basis for \( \text{Null}(A^T) \): \( \{ (-1, 1) \} \)

Check: \( (1, -1, 1) \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = (0, 0, 0) \)

so at least \( (-1, 1) \in \text{Null}(A^T) \) ✓

Consequence:

\[ \dim \text{Null}(A^T) = m - r = \# \text{rows} - \text{rank} \]

Proof of the Thm: Suppose \( A \xrightarrow{\text{REF}} U \). Then

\[ U = E \cdot A \quad E = \text{product of elementary matrices} \]

\[ \Rightarrow E \cdot (A | I_m) = (EA | EI_m) = (U | E) \]
So the result of performing elimination on \((A|I_m)\) is \((U|E)\).

If \(U\) is in REF and the last \(m-r\) rows are zero then we claim:

\[
\text{Null}(U^T) = \text{Span}\{e_{r+1}, e_{r+2}, \ldots, e_m\}.
\]

We know \(U^Te_i = \text{the \(i\)th row of } U\).

We know from before that the nonzero rows of \(U\) are LI. So if \((x_1, \ldots, x_m) \in \text{Null}(U^T)\) then

\[
0 = U^T(x_m) = x_1U^Te_1 + \cdots + x_rU^Te_r
\]

\[
+ x_{r+1}U^Te_{r+1} + \cdots + x_mU^Te_m.
\]

These are 0 because the last \(m-r\) rows of \(U\) are 0.

\[
= x_1U^Te_1 + \cdots + x_rU^Te_r = 0
\]

This implies \(x_1 = \cdots = x_r = 0\) because the first \(r\) rows of \(U\) are LI.

So \(0 = U^T(x_1, \ldots, x_m) \iff x_1 = \cdots = x_r = 0 \iff (x_1, \ldots, x_m) \in \text{Span}\{e_{r+1}, e_{r+2}, \ldots, e_m\}\).
This proves the claim.

Now, \( U = EA \Rightarrow U^T = A^TE^T \), so
\[
A^TE^T x = 0 \Rightarrow U^T x = 0
\]

\[
\Leftrightarrow x = a_1 E^T e_1 + a_2 E^T e_2 + \ldots + a_m E^T e_m
\]

But \( E^T e_i \) is the \( i \)th row of \( E \), so
\[
E^T x = a_1 E^T e_1 + a_2 E^T e_2 + \ldots + a_m E^T e_m
\]

\[
= \text{ a LC of the last } m-r \text{ rows of } E
\]

so \( A^T E^T x = 0 \)
\[
\Leftrightarrow E^T x \in \text{Span } \{ \text{last } m-r \text{ rows of } E \}.
\]

(I've left out some details at the end)

NB: The left null space is changed by row operations:

\[
A = \begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1 \\
1 & 2 & -1 & -2
\end{bmatrix}
\]

\[
\text{Null}(A^T) = \text{Span } \{(1, -1, 1)\}
\]

\[
U = \begin{bmatrix}
1 & 0 & 2 & 2 & 1 \\
0 & 0 & 2 & -3 & -3
\end{bmatrix}
\]

\[
\text{Null}(U^T) = \text{Span } \{(0, 0, 0, 1)\}
\]
Summary: Four Subspaces

A: an $m \times n$ matrix of rank $r$

<table>
<thead>
<tr>
<th>Subspace</th>
<th>of</th>
<th>row/column</th>
<th>dim</th>
<th>basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Col(A)</td>
<td>$\mathbb{R}^m$</td>
<td>col</td>
<td>$r$</td>
<td>pivot cols of A</td>
</tr>
<tr>
<td>Nul(A)</td>
<td>$\mathbb{R}^n$</td>
<td>row</td>
<td>$n-r$</td>
<td>vectors in PVF</td>
</tr>
<tr>
<td>Row(A)</td>
<td>$\mathbb{R}^n$</td>
<td>row</td>
<td>$r$</td>
<td>nonzero rows of REF</td>
</tr>
<tr>
<td>Nul(Aᵀ)</td>
<td>$\mathbb{R}^m$</td>
<td>col</td>
<td>$m-r$</td>
<td>last $m-r$ rows of E</td>
</tr>
</tbody>
</table>

The row picture subspaces ($\text{Nul}(A), \text{Row}(A)$) are unchanged by row operations.

The col picture subspaces ($\text{Col}(A), \text{Nul}(Aᵀ)$) are changed by row operations.

\[ \dim \text{Row}(A) + \dim \text{Nul}(A) = n \]
\[ \dim \text{Col}(A) + \dim \text{Nul}(Aᵀ) = m \]
Consequences:

\[
\text{Row Rank} = \text{Column Rank}
\]
\[
\dim \text{Row}(A) = \text{rank} = \dim \text{Col}(A)
\]

So \(A\) & \(A^T\) have the same # pivots — in completely different positions! (#W#5)

\[
\text{Rank - Nullity}
\]
\[
\dim \text{Col}(A) + \dim \text{Null}(A) = n = \#\text{cols}
\]
\[
\dim \text{Row}(A) + \dim \text{Null}(A^T) = m = \#\text{rows}
\]

[\text{demos}]

NB: You can compute bases for all four subspaces by doing elimination once.

\[
A \rightarrow [A | \text{Im}] \rightarrow [\text{RREF(A)} | \text{E}]
\]

- Get the pivots of \(A\) \(\rightarrow\) \(\text{Col}(A)\)
- Get \(\text{RREF(A)}\) \(\rightarrow\) PVE of \(A\text{x}=0 \rightarrow \text{Null}(A)\)
- Get nonzero rows of \(\text{RREF(A)} \rightarrow \text{Row(A)}\)
- Get rows of \(E \rightarrow \text{Null}(A^T)\)
Full-Rank Matrices

A "random" matrix will have largest rank possible. This is an important special case.

Def: An $m \times n$ matrix $A$ of rank $r$ has:

- **full column rank if** $r=n$, eg. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
  (every column has a pivot)
- **full row rank if** $r=m$, eg. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
  (every row has a pivot)

NB: Each row & column has at most one pivot so $r \leq \min \{m,n\}$

Hence full row/column rank means full rank, i.e. largest possible rank.

NB: $A$ has full column rank $\Rightarrow n=r \leq m$

$\Rightarrow A$ is tall (at least as many rows as cols)

$A$ has full row rank $\Rightarrow m=r \leq n$

$\Rightarrow A$ is wide (at least as many cols as rows)
We've seen several properties of matrices that translate into "there's a pivot in every column".

**Thm:** The Following Are Equivalent (TFAE):

(for a given matrix $A$, all are true or all are false)

1. $A$ has **full column rank**
   1'. $A$ has a pivot in every column
   1'' $A$ has no free columns.
2. $\text{Null}(A) = \{0\}$
   2'. $Ax = 0$ has only the trivial solution.
   2'' $Ax = b$ has 0 or 1 soln for every $b \in \mathbb{R}^n$
3. The columns of $A$ are LI
4. $\dim \text{Col}(A) = n$
5. $\dim \text{Row}(A) = n$
   5' $\text{Row}(A) = \mathbb{R}^n$

**NB:** (5) $\iff$ (5') because:

The only $n$-dimensional subspace of $\mathbb{R}^n$ is all of $\mathbb{R}^n$

**Eg:** There is no plane in $\mathbb{R}^2$ that doesn't fill up all of $\mathbb{R}^2$. 
We've seen several properties of matrices that translate into "there's a pivot in every row".

**Thm: TFAE:**

1. \( A \) has full row rank
   - \( A \) has a pivot in every row
   - \( A \) RER of \( A \) has no zero rows

2. \( \dim \text{Col}(A) = m \)
   - \( \dim \text{Col}(A) = \mathbb{R}^m \)
   - \( Ax = b \) is consistent for every \( b \in \mathbb{R}^m \)
     (has 1 or \( \infty \) solutions)

3. The columns of \( A \) span \( \mathbb{R}^m \)

4. \( \dim \text{Row}(A) = m \)

5. \( \text{Nul}(A^T) = \{0\} \)

Again, (2) \( \iff \) (2') because the only \( m \)-dimensional subspace of \( \mathbb{R}^m \) is all of \( \mathbb{R}^m \).
If \( A \) has full column rank and full row rank then \( n = r = m \)

\[ \Rightarrow A \] is square and has \( n \) pivots: invertible.

Thm: For an \( n \times n \) matrix \( A \), TFAE:

1. \( A \) is invertible
2. \( A \) has full column rank
3. \( A \) has full row rank
4. \( \text{RREF}(A) = I_n \)
5. There is a matrix \( B \) with \( AB = I_n \)
6. There is a matrix \( B \) with \( BA = I_n \)
7. \( Ax = b \) has exactly one solution for every \( b \) \( \Leftrightarrow \) namely, \( x = A^{-1}b \)
8. \( A^T \) is invertible

\( \Leftrightarrow \) (row rank = col rank)
Consequence: Let \( \{v_1, \ldots, v_n\} \) be vectors in \( \mathbb{R}^n \)
\[ \implies A = (v_1, \ldots, v_n) \] is an \( n \times n \) matrix.

(1) \( \text{Span}\{v_1, \ldots, v_n\} = \mathbb{R}^n \iff \text{Col}(A) = \mathbb{R}^n \)
\[ \iff A \text{ has FRR} \]
\[ \iff A \text{ is invertible} \]

(1) \( \{v_1, \ldots, v_n\} \) is LI
\[ \iff Ax = 0 \text{ has only the trivial soln} \]
\[ \iff A \text{ has FCR} \]
\[ \iff A \text{ is invertible} \]

Of course, (1)+(2) means \( \{v_1, \ldots, v_n\} \) is a basis for \( \mathbb{R}^n \), so

( basis for ) \[ \begin{bmatrix} \mathbb{R}^n \end{bmatrix} \] \[ \equiv \begin{bmatrix} \text{columns of an} \\
\text{invertible } n \times n \text{ matrix} \end{bmatrix} \]

More on this next time (Basis Theorem).