Welcome to Math 218D-1!

Introduction to Linear Algebra

What is Linear Algebra?
The study of (systems of) linear equations

Like: \[ y = 3x + 2 \quad \rightarrow \quad -3x + y = 2 \]
(usually put variables on the left & constants on the right)

Or: \[
\begin{align*}
x + y + z &= 1 \\
y - z &= -3
\end{align*}
\]
\[ \uparrow \quad \uparrow \quad \uparrow \] (arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number) \cdot (variable) or (number)

Note: \[ xy + z = 1 \quad \text{\textasciitilde product of variables} \]
\[ x + 3 = y^2 \quad \text{\textasciitilde power of a variable} \]
\[ e^x = \cos(y) \quad \text{\textasciitilde complicated functions} \]

Why linear algebra?
- It's simple enough to understand very well & program computers to do quickly.
It's powerful enough to solve a huge range of different problems.

Eg:

Here's a map of roads in the town square:

- Question: How many cars/hr travel on the unlabeled roads?

Step 0: When you have an unknown quantity, give it a name!

Observation:

# cars entering each intersection = # cars leaving if

\[
\begin{align*}
A: & \quad 120 + w = 250 + x \\
B: & \quad 120 + x = 70 + y \\
C: & \quad 530 + y = 390 + z \\
D: & \quad 115 + z = 175 + w
\end{align*}
\]

This is a system of 4 linear equations in 4 unknowns!
Question: You know a priori that there are infinitely many solutions. How?

Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations. It is your job to turn your question into a linear algebra problem (that a computer can solve) and interpret the answer.

Eg: An asteroid has been observed at coordinates: (0, 2), (2, 1), (-1, -1), (-1, -2), (3, 1), (-1, 1)

Question: What is the most likely orbit? Will the asteroid crash into the Earth?

Fact: The orbit is an ellipse.

Equation for an ellipse:

\[ x^2 + By^2 + Cxy + Dx + Ey + F = 0 \]

Wait! Isn't this a nonlinear equation? ...
For our points to lie on the ellipse, substitute the coordinates into \((x,y)\) as these should hold:

\[
\begin{align*}
(0,2): & \quad 0 + 4B + 2C + 2D + E + F = 0 \\
(2,1): & \quad 4 + B + 2C + 2D + E + F = 0 \\
(1,-1): & \quad 1 + B - C + D - E + F = 0 \\
(-1,-2): & \quad 1 + 4B + 2C - D - 2E + F = 0 \\
(-3,1): & \quad 9 + B - 3C + D - 3E + F = 0 \\
(-1,1): & \quad 1 + B - C - D + E + F = 0
\end{align*}
\]

This is a system of six linear equations in 5 variables.

Note: The variables are the coefficients \(B,C,D,E,F\).

Remember, were finding the equation of the ellipse.

NB: There is no solution — the points do not lie on an ellipse (perhaps due to measurement error).

Question: What is the best approximate solution?

Answer: [demo]
Historical note: Gauss invented much of what you'll learn to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on demos: I created these to help give you a geometric understanding of linear algebra.

→ It took a lot of work.
→ Really, it was hard.
→ Why would I do that? I want you to have a geometric understanding.

Upshot: Play with the demos! Don't turn off your brain when we do geometry! You will be expected to draw pictures on exams!

Eg: In a population of rabbits,

1) Half survive their first year.
2) Half of those survive their second year.
3) The maximum life span is 3 years.
4) Each rabbit produces (on average) 0, 6, 8 offspring in years 0, 1, 2, respectively.

Question: How many rabbits will there be in 100 years?
Step 0: Give names to the unknowns

\[ X_n: \# \text{rabbits aged 0 in year } n \]
\[ Y_n: \# \text{rabbits aged 1 in year } n \]
\[ Z_n: \# \text{rabbits aged 2 in year } n \]

Rules: \[ X_{2021} = 6Y_{2020} + 8Z_{2020} \]
\[ Y_{2021} = \frac{1}{2}X_{2020} \]
\[ Z_{2021} = \frac{1}{2}Y_{2020} \]

A system of equations of this form is called a difference equation. We'll solve them using eigenvalues & diagonalization (week 10).

**[demo]** It looks like eventually,

- The population doubles each year
- The ratio of rabbits aged 0 : 1 : 2 is \( \approx 16 : 4 : 1 \)

Comes from: \( \begin{pmatrix} \frac{1}{2} & 0 & 8 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \) w/eigenvalue \( 2 \).

Other examples:
- Google PageRank lets you search the Web with a Markov chain — a special type of difference equation.
• Netflix knows what movies you'll like using the Singular Value Decomposition (weeks 14-15).

Geometry of Solutions

Convention: given a system of linear equations, put the constant term on the right of the =, and put the variables on the left, organized in columns.

\[
\begin{align*}
120 + w &= 250 + x \\
120 + x &= 70 + y \\
570 + y &= 390 + z \\
115 + z &= 175 + w
\end{align*}
\]

Def: The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

Question: What does the solution set of a system of linear equations look like?
One equation in 2 variables:
\[ x - 2y = 1 \implies y = \frac{1}{2}x - \frac{1}{2} \]

One equation in 3 variables:
\[ x + ty + z = 1 \implies z = 1 - x - y \]

One equation in 4 variables: “3-plane in 4-space”

Note on dimensions: Students often want to say “the fourth dimension is time.” Einstein used \( \mathbb{R}^4 \) (4-space) to model spacetime, but it models lots of other things too. (like traffic around the town square…)

2 equations in 2 variables:
\[ \begin{align*}
  x - 2y &= 1 \\
  3x + 2y &= 11
\end{align*} \]

Where are both true?
Intersection of 2 lines.
(answer: \((3, 1)\) )
What else can happen?

\[
\begin{align*}
    x - 2y &= 1 \\
    3x - 6y &= 3 \\
\end{align*}
\]

Same line: \( \infty \) solutions.

\[
\begin{align*}
    x - 2y &= 1 \\
    3x - 6y &= 6 \\
\end{align*}
\]

Parallel lines: 0 solutions

2 equations in 3 variables:

\[
\begin{align*}
    x + y + z &= 1 \\
    x - z &= 0 \\
\end{align*}
\]

Intersection of two planes in space \([\text{dune}]\)

In this case, it's a line.

3 equations in 3 variables:

\[
\begin{align*}
    x + y + z &= 1 \\
    x - z &= 0 \\
    y &= 0 \\
\end{align*}
\]

\[
\begin{align*}
    x &= v_2 \\
    y &= 0 \\
    z &= v_2 \\
\end{align*}
\]

Intersection of three planes in space: in this case it's a point.
Question: How many "ways" can 3 planes in space intersect?
Answer: 8

Syllabus Stuff: see the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:
  https://services.math.duke.edu/~jdr/2324s-218/

- We will use Sakai for:
  - Announcements
  - Gradebook
  - Gradescope

  !! Please use the Gradescope tab on Sakai instead of going to gradescope.com.

  - Ed Discussion: for asking questions (replaces Piazza).

  !! Don't email us w/math questions! Post it here instead - then everyone sees it & anyone can answer.

  - WarpWire (see below)
Textbook:
- Strang, "Introduction to Linear Algebra," 5th ed. We'll only follow this loosely. Also see
- Margalit-Rabinof, “Interactive Linear Algebra” (on the course website). You'll get a link to a beta version aimed just at this course!

Quizzes: a 10-minute small-group quiz will be held at the beginning of each discussion section. It's very basic — just tests if you've looked over your notes.

Homework: due Wednesday 11:59 pm every week.
- Meant to be long and hard: you need practice to learn math, and practice takes time.
- Scan & submit on Gradescope. Use a scanning app!
- Tag the pages on Gradescope with the problems on that page!

Midterms: 2 of them, during discussion slots.
Final: as scheduled by the registrar.
Help! • Come to office hours!
  • Ask on Ed Discussion
  • See course webpage.

Recorded Lecture:
  Basics of vector & matrix algebra.
  Watch before Tuesday (on Warp Wine)
  HW#1 covers that material.