Math 218D-1: Homework #9

due Wednesday, March 20, at 11:59pm

1. **a)** Compute the determinants of the matrices in HW8#13 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!

b) Compute the determinants of the matrices in HW8#13(b) and (d) *again* using Sarrus' scheme.

c) For the matrix of HW8#13(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus' scheme. Did you get the determinant?

2. Compute

$$\det\left[\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} - \lambda I_3\right]$$

where λ is an unknown real number. Your answer will be a function of λ.

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Compute the cofactor matrix C of A.

b) Compute AC^T. What is the relationship between C^T and A^{-1}?

4. Consider the $n \times n$ matrix F_n with 1's on the diagonal, 1's in the entries immediately below the diagonal, and -1's in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \cdots.$$

a) Show that $\det(F_2) = 2$ and $\det(F_3) = 3$.

b) Expand in cofactors to show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$.

c) Compute $\det(F_4), \det(F_5), \det(F_6), \det(F_7)$ using **b)**.

This shows that $\det(F_n)$ is the nth *Fibonacci number*. (The sequence usually starts with $1, 1, 2, 3, \ldots$, so our $\det(F_n)$ is the usual $n + 1$st Fibonacci number.)

5. Let A be an $n \times n$ invertible matrix with integer (whole number) entries.

a) Explain why $\det(A)$ is an integer.

b) If $\det(A) = \pm 1$, show that A^{-1} has integer entries.

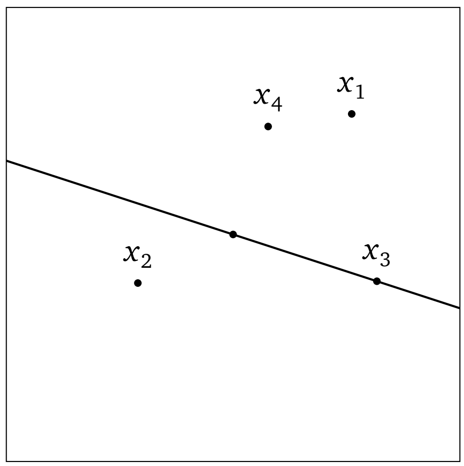
c) If A^{-1} has integer entries, show that $\det(A) = \pm 1$.

6. Let V be a subspace of $\mathbf{R}^n$. The matrix for *reflection over* V is

$$R_V = I_n - 2P_{V^\perp},$$

where $P_{V^\perp} = I_n - P_V$ is the projection matrix onto $V^\perp$.

a) Suppose that V is the line in the picture. Draw the vectors $R_V x_1, R_V x_2, R_V x_3$, and $R_V x_4$ as points in the plane.
[**Hint:** First draw $-2(x_i)_{V^\perp}$.]



b) Show that any reflection matrix R_V is orthogonal.
[**Hint:** Recall that $P_{V^\perp}^2 = P_{V^\perp} = P_{V^\perp}^T$.]

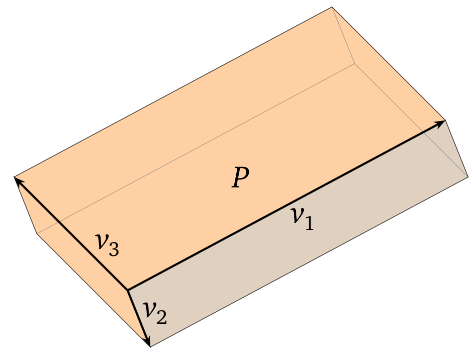
c) Let V be the plane $x + y + z = 0$. Compute R_V and $\det(R_V)$.

More generally, any reflection over a plane in $\mathbf{R}^3$ has determinant -1.

d) Now compute $\det(R_L)$, where L is the x-axis in $\mathbf{R}^3$.

7. Consider the paralellepiped P in $\mathbf{R}^3$ spanned by

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$



a) Compute the volume of P using a triple product $(v_1 \times v_2) \cdot v_3$.

b) Compute the area of each face of P using cross products.

c) If the "base" of P is the paralellogram spanned by v_1 and v_2 (blue in the picture), show that the height of P is $\|v_3\| \sin\theta$, where θ is the angle that v_3 makes with the base. (Draw a simpler picture.)

d) The volume of P is the area of the base of P times its height. How do you reconcile **c)** with **a)**? (Remember that $\|u \cdot v\| = \|u\|\,\|v\| \cos(\text{the angle from } u \text{ to } v)$.)

8. Use a cross product to find an implicit equation for the plane

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

Compare HW6#16(a).

9. **a)** Let $v = (a, b)$ and $w = (c, d)$ be vectors in the plane, and let $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$, explain how the right-hand rule determines the sign of $\det(A)$.

b) Using the identity

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

explain how the right-hand rule determines the sign of a 3×3 determinant.

10. Decide if each statement is true or false, and explain why.

a) The determinant of the cofactor matrix of A equals the determinant of A.

b) $u \times v = v \times u$.

c) If $u \times v = 0$ then $u \perp v$.

11. For each matrix A and each vector v, decide if v is an eigenvector of A, and if so, find the eigenvalue λ.

$$\textbf{a)}\ \begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\textbf{c)}\ \begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\textbf{e)}\ \begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

12. For each matrix A and each number λ, decide if λ is an eigenvalue of A; if so, find a basis for the λ-eigenspace of A.

$$\textbf{a)}\ \begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \ \lambda = 1 \qquad \textbf{b)}\ \begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \ \lambda = -1$$

$$\textbf{c)}\ \begin{pmatrix} 2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix} \ \lambda = 3 \qquad \textbf{d)}\ \begin{pmatrix} 2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix} \ \lambda = 2$$

$$\textbf{e)}\ \begin{pmatrix} 3 & 1 & -2 \\ -2 & 0 & 4 \\ -1 & -1 & 4 \end{pmatrix} \ \lambda = 2 \qquad \textbf{f)}\ \begin{pmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{pmatrix} \ \lambda = 0$$

$$\textbf{g)}\ \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \ \lambda = 7 \qquad \textbf{h)}\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \lambda = 0$$

13. Suppose that A is an $n \times n$ matrix such that $Av = 2v$ for some $v \neq 0$. Let C be any invertible matrix. Consider the matrices

a) A^{-1} **b)** $A + 2I_n$ **c)** A^3 **d)** CAC^{-1}.

Show that v is an eigenvector of **a)–c)** and that Cv is as eigenvector of **d)**, and find the eigenvalues.

14. Here is a handy trick for computing eigenvectors of a 2×2 matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with eigenvalue λ. Explain why $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ and $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$ are λ-eigenvectors of A if they are nonzero.

[**Hint:** If λ is an eigenvalue of A, then the rows of $A - \lambda I_2$ are linearly dependent.]

For which matrices A does this trick fail?

15. **a)** Show that A and A^T have the same eigenvalues.

b) Give an example of a 2×2 matrix A such that A and A^T do not share any eigenvectors.

c) A *stochastic matrix* is a matrix with nonnegative entries such that the entries in each column sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix. [**Hint:** show that $(1, 1, \ldots, 1)$ is an eigenvector of A^T.]

16. **a)** Find all eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 & -5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

b) Explain how to find the eigenvalues of any triangular matrix.

17. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns. Prove that any (real) eigenvalue of an orthogonal matrix Q is ± 1.

18. Give an example of each of the following, or explain why no such example exists.

a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.

b) A 2×2 orthogonal matrix with no real eigenvalues.

19. Suppose that A is a square matrix such that A^k is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of A.

20. Decide if each statement is true or false, and explain why.

a) If v, w are eigenvectors of a matrix A and $v + w \neq 0$, then $v + w$ is also an eigenvector.

b) An eigenvalue of $A + B$ is the sum of an eigenvalue of A and an eigenvalue of B.

c) An eigenvalue of AB is the product of an eigenvalue of A and an eigenvalue of B.

d) If $Ax = \lambda x$ for some vector x, then λ is an eigenvalue of A.

e) A matrix with eigenvalue 0 is not invertible.

f) The eigenvalues of A are equal to the eigenvalues of a row echelon form of A.