

Math 218D-1: Homework #8

due Wednesday, March 6, at 11:59pm

1. For each set of vectors, decide if they are orthogonal, orthonormal, or neither; then compute $Q^T Q$, where Q is the matrix with the vectors as columns.

$$\begin{array}{ll} \text{a) } \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\} & \text{b) } \left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} \\ \text{c) } \left\{ \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\} & \text{d) } \left\{ \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \right\} \end{array}$$

2. The following subspaces V are given as the span of an *orthogonal* set of vectors. For each subspace V and vector b , compute the orthogonal projection b_V using the *projection formula*, and compute the projection matrix P_V using the *outer product formula*.

$$\begin{array}{ll} \text{a) } V = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} & b = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \\ \text{b) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} & b = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \\ \text{c) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\} & b = \begin{pmatrix} 4 \\ 2 \\ -4 \\ 2 \end{pmatrix} \\ \text{d) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} & b = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix} \\ \text{e) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \right\} & b = \begin{pmatrix} 9 \\ -2 \\ 3 \end{pmatrix} \end{array}$$

3. For each subspace V of Problem 2, scale the spanning vectors to find an *orthonormal* basis of V , and (re)compute the projection matrix P_V using the formula $P_V = QQ^T$. (Your answers should be exact, in terms of square roots.)
4. Suppose that $\{u_1, u_2, \dots, u_n\}$ is an orthonormal basis of \mathbf{R}^n . Use the outer product formula to explain why

$$I_n = u_1 u_1^T + u_2 u_2^T + \cdots + u_n u_n^T.$$

5. Give an example of each of the following, or explain why no such example exists.
- A matrix Q with orthonormal columns, but $QQ^T \neq I_n$.
 - Two nonzero orthogonal vectors that are linearly dependent.
 - An orthonormal basis for the plane $x + y + z = 0$.

6. Use the Gram–Schmidt process to find orthogonal bases of the following subspaces.

$$\begin{array}{ll} \text{a) } \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} & \text{b) } \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \\ -1 \end{pmatrix} \right\} \\ \text{c) } \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\} & \text{d) } \text{Nul} \begin{pmatrix} 1 & 2 & -1 & 4 \\ 3 & 6 & -3 & 12 \end{pmatrix} \end{array}$$

7. Consider the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

and the vector

$$v = \begin{pmatrix} 4 \\ -1 \\ 3 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \in V.$$

Find all vectors **contained in** V that are orthogonal to v .

[**Hint:** apply Gram–Schmidt to a set containing v .]

8. Consider the subspace $V = \text{Col}(A)$, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & 4 \\ -2 & -1 & -1 \end{pmatrix}.$$

Find an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of \mathbf{R}^4 such that $\{u_1, u_2, u_3\}$ is a basis for V . Your answer should be exact, in terms of square roots.

9. For each of the following matrices A and vectors b , find the QR decomposition of A , and find the least-squares solution of $Ax = b$ by back-substitution in $R\hat{x} = Q^T b$. Your answers should be exact, in terms of square roots.

$$\text{a) } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & 4 \\ -2 & -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

10. In this problem, we use a QR decomposition to *quickly* compute the best-fit parabola with specified y -values at $x = -2, -1, 1, 2$, as in HW7#12.

- a) Compute the matrix A such that the least squares solution of $A(C, D, E) = (b_1, b_2, b_3, b_4)$ gives the coefficients of the parabola $y = Cx^2 + Dx + E$ that best fits the data points $(-2, b_1), (-1, b_2), (1, b_3), (2, b_4)$. (Presumably you computed this in HW7#12.)
- b) Find the QR decomposition of A .
- c) Find the best-fit parabola through the points $(-2, 3), (-1, -1), (1, 1), (2, 3)$ by back-substitution in $R\hat{x} = Q^T b$. You should get the same answer as in HW7#12.

Note that we can now repeat part c) with new y -values in $O(n^2)$ time.

11. Recall that an *orthogonal* matrix is a square matrix with *orthonormal* columns.

- a) If Q is an orthogonal matrix, show that Q^{-1} is orthogonal.
- b) If Q_1 and Q_2 are orthogonal matrices of the same size, show that $Q_1 Q_2$ is orthogonal.

12. Decide if each statement is true or false, and explain why.

- a) A matrix with orthogonal columns has full row rank.
- b) If $\{v_1, \dots, v_n\}$ is a linearly independent set of vectors, then it is orthogonal.
- c) If $\{v_1, v_2\}$ is a basis for a plane V , then for any vector b ,

$$b_V = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} v_2.$$

- d) If Q has orthonormal columns, then the distance from x to y equals the distance from Qx to Qy .
- e) If $A = QR$ is a QR-factorization of a matrix A , then the rows of Q form an orthonormal basis for $\text{Row}(A)$.

13. Compute the determinants of the following matrices *using Gaussian elimination*.

$$\begin{array}{ll} \text{a) } \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} & \text{b) } \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} \\ \text{c) } \begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} & \text{d) } \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \end{array}$$

14. Suppose that

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10.$$

Find the determinants of the following matrices.

$$\begin{array}{lll} \text{a)} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} & \text{b)} \begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix} & \text{c)} \begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix} \\ \text{d)} \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} & \text{e)} \begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix} & \text{f)} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \\ \text{g)} 2 \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \text{h)} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \text{i)} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \\ \text{j)} - \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \text{k)} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^3 & \text{l)} \begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix} \end{array}$$

15. Find $\det(E)$ when:

- a) E is the elementary matrix for a row replacement.
- b) E is the elementary matrix for $R_i \times = c$.
- c) E is the elementary matrix for a row swap.

16. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

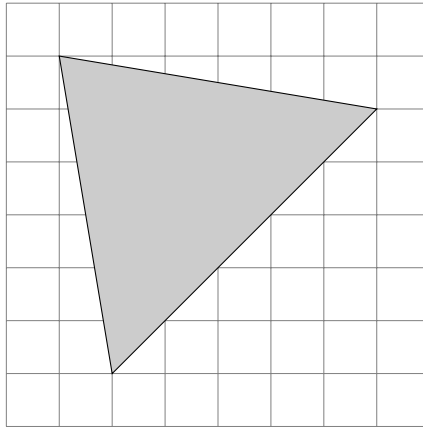
What is $\det(A)$?

17. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns, or equivalently, a square matrix Q such that $Q^T Q = I_n$. Prove that every orthogonal matrix has determinant ± 1 .

18. Let V be a subspace of \mathbf{R}^n and let P_V be the projection matrix onto V .

- a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.
- b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

19. Let A be an $n \times n$ matrix with columns v_1, v_2, \dots, v_n .
- Show that if $\{v_1, v_2, \dots, v_n\}$ is orthogonal then $|\det(A)| = \|v_1\| \|v_2\| \cdots \|v_n\|$.
[Hint: Compute $A^T A$ and its determinant.]
 - Suppose that A is invertible. Show that $|\det(A)| \leq \|v_1\| \|v_2\| \cdots \|v_n\|$, with equality if and only if the set $\{v_1, v_2, \dots, v_n\}$ is orthogonal.
[Hint: Use HW6#14(c) and the QR decomposition of A .]
20. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



21. Decide if each statement is true or false, and explain why.
- $\det(A + B) = \det(A) + \det(B)$.
 - $\det(ABC^{-1}) = \frac{\det(A)\det(B)}{\det(C)}$.
 - $\det(AB) = \det(BA)$.
 - $\det(3A) = 3 \det(A)$.
 - If A^5 is invertible then A is invertible.
 - The determinant of A is the product of its diagonal entries.
 - If the columns of A are linearly dependent, then $\det(A) = 0$.
 - If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.