Math 218D-1: Homework #7

Answer Key

1. For each column space $V$, compute the projection matrix $P_V$. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

   a) $V = \text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$
   b) $V = \text{Col} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 0 \end{pmatrix}$
   c) $V = \text{Col} \begin{pmatrix} 2 & -1 \\ -5 & 5 \\ 6 & 12 \end{pmatrix}$

Solution.

   a) $\frac{1}{9} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -2 & 8 \end{pmatrix}$
   b) $\frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -1 & 1 \\ 1 & 3 \end{pmatrix}$
   c) $\frac{1}{195} \begin{pmatrix} 26 & -65 & 13 \\ -65 & 170 & 5 \\ 13 & 5 & 194 \end{pmatrix}$

2. For each subspace $V$, compute the projection matrix $P_V$. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

   a) $V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$
   b) $V = \text{Nul} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Solution.

   a) $\frac{1}{5} \begin{pmatrix} 1 & 0 \\ 0 & 5 \\ 2 & 0 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$
   b) $\frac{1}{5} \begin{pmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

3. For each vector $v$, compute the projection matrix onto $V = \text{Span}\{v\}$ using the formula $P_V = vv^T / v \cdot v$.

   a) $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
   b) $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
   c) $v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ (in $\mathbb{R}^n$)

Solution.
4. Compute $P_V$ and $P_{V^\perp}$ when $V$ is the column space of an invertible $n \times n$ matrix.

Solution.
If $A$ is invertible then $V = \text{Col}(A) = \mathbb{R}^n$, so $P_V = I_n$ and $P_{V^\perp} = I_n - I_n = 0$.

5. For each subspace $V$, compute the projection matrix $P_V$.
   a) $\{(x, y, x) : x, y \in \mathbb{R}\}$.
   b) $\{(x, y, z) \in \mathbb{R}^3 : x = 2y + z\}$.
   c) The solution set of the system of equations \[ \begin{align*}
   x + y + z &= 0 \\
   x - 2y - z &= 0.
   \end{align*} \]
   d) $\{x \in \mathbb{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\
   1 & 0 & 0 \\
   0 & 1 & 0 \end{pmatrix}$.
   e) The subspace of all vectors in $\mathbb{R}^3$ whose coordinates sum to zero.
   f) The intersection of the plane $x - 2y - z = 0$ with the $xy$-plane.
   g) The line $\{(t, -t, t) : t \in \mathbb{R}\}$.

[Hint: Compare HW5#9 and HW6#7. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]

Solution.
   a) $\frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\
   0 & 2 & 0 \\
   1 & 0 & 1 \end{pmatrix}$
   b) $\frac{1}{6} \begin{pmatrix} 5 & 2 & 1 \\
   2 & 2 & -2 \\
   1 & -2 & 5 \end{pmatrix}$
   c) $\frac{1}{14} \begin{pmatrix} 1 & 2 & -3 \\
   2 & 4 & -6 \\
   -3 & -6 & 9 \end{pmatrix}$
   d) $\frac{1}{273} \begin{pmatrix} 256 & 64 & 16 \\
   64 & 16 & 4 \\
   16 & 4 & 1 \end{pmatrix}$
   e) $\frac{1}{3} \begin{pmatrix} 1 & -1 & 1 \\
   -1 & 2 & -1 \\
   -1 & -1 & 2 \end{pmatrix}$
   f) $\frac{1}{5} \begin{pmatrix} 4 & 2 & 0 \\
   2 & 1 & 0 \\
   0 & 0 & 0 \end{pmatrix}$
   g) $\frac{1}{3} \begin{pmatrix} 1 & -1 & 1 \\
   -1 & 1 & -1 \\
   1 & -1 & 1 \end{pmatrix}$

6. Consider the matrix
   \[ A = \begin{pmatrix} 1 & 2 \\
   -1 & 1 \\
   2 & 2 \\
   4 & 3 \end{pmatrix}, \]
   and let $V = \text{Col}(A)$. 

}\text{Answer Key}{
a) Compute $P_V$ using the formula $P_V = A(A^T A)^{-1} A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \text{Nul}(A^T)$.

c) Let $B$ be the matrix with columns $v_1, v_2$, and compute $P_{V^\perp}$ using the formula $B(B^T B)^{-1} B^T$.

d) Verify that your answers to (a) and (c) sum to $I_4$.

This illustrates the fact that once you’ve computed $P_V$, there’s no need to compute $P_{V^\perp}$ separately. It’s a lot of extra work!

**Solution.**

a) $P_V = \frac{1}{107} \begin{pmatrix} 38 & 43 & 22 & 17 \\ 43 & 74 & 8 & -23 \\ 22 & 8 & 24 & 38 \\ 17 & -23 & 38 & 78 \end{pmatrix}$

b) $\left\{ \begin{pmatrix} -4/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7/3 \\ 5/3 \\ 0 \\ 1 \end{pmatrix} \right\}$

c) $P_{V^\perp} = \frac{1}{107} \begin{pmatrix} 69 & -43 & -22 & -17 \\ -43 & 33 & -8 & 23 \\ -22 & -8 & 83 & -38 \\ -17 & 23 & -38 & 29 \end{pmatrix}$

7. Compute the matrices $P_1, P_2$ for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute $P_1 P_2$, and explain why it is what it is.

**Solution.**

\[ P_1 = \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \quad P_2 = \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \quad P_1 P_2 = 0 \]

Evaluating $P_1 P_2 x$ means first orthogonally projecting $x$ onto the line through $a_1$, then orthogonally projecting the result onto the line through $a_2$. But $a_1$ and $a_2$ are orthogonal.

8. Consider the plane $V$ defined by the equation $x + 2y - z = 0$. Compute the matrix $P_V$ for orthogonal projection onto $V$ in two ways:

a) Find a basis for $V$, put your basis vectors into a matrix $A$, and use the formula $P_V = A(A^T A)^{-1} A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $v v^T / v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.
If $V$ is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

Solution.

a) We can take

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad P_V = \frac{1}{6} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

b) Taking \( v = (1, 2, -1) \), we have

$$P_{V\perp} = \frac{1}{6} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} \quad \Rightarrow \quad P_V = I_3 - P_{V\perp} = \frac{1}{6} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

The second method is much faster, as it does not involve Gauss–Jordan elimination or inversion.

9. Decide if each statement is true or false, and explain why. In each statement, \( V \) is a subspace of \( \mathbb{R}^n \).

a) The rank of \( P_V \) is equal to \( \text{dim}(V) \).

b) \( P_V P_{V\perp} = 0 \).

c) \( P_V + P_{V\perp} = 0 \).

d) \( \text{Col}(P_V) = V \).

e) \( \text{Nul}(P_V) = V \).

f) \( \text{Row}(P_V) = \text{Col}(P_V) \).

g) \( \text{Nul}(P_V)^\perp = \text{Col}(P_V) \).

Solution.

a) True.

b) True.

c) False.

d) True.

e) False.

f) True.

g) True.
10. Find all least-squares solutions $\hat{x}$ of each of the following systems of equations $Ax = b$, and compute the projection $b_V$ of $b$ onto $V = \text{Col}(A)$ and the minimum value of $\|A\hat{x} - b\|$.

a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$  
   $b) \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$  
   c) $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} x = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$  
   d) $\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

Solution.

a) $\hat{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  
   $b_V = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$  
   $\|A\hat{x} - b\| = 3$

b) $\hat{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  
   $b_V = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 5 \end{pmatrix}$  
   $\|A\hat{x} - b\| = 4$

c) $\hat{x} = \begin{pmatrix} -\frac{3}{5} \\ 2 \end{pmatrix}$  
   $b_V = \begin{pmatrix} 7 \\ -19 \\ -4 \end{pmatrix}$  
   $\|A\hat{x} - b\| = 195$

d) $\hat{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$  
   $b_V = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$  
   $\|A\hat{x} - b\| = 0$

11. Consider the data points $p_1 = (1, 2), p_2 = (2, -1), p_3 = (3, 0), p_4 = (4, 5)$.

a) Find the best-fit line $y = Cx + D$ through these four points, and draw it on the grid below.
b) For each data point \( p_i = (a_i, b_i) \), draw the error bar from \((a_i, y(a_i))\) to \((a_i, b_i)\).

c) What is the minimum value of \( \sum_{i=1}^{4} (b_i - y(a_i))^2 \)? How do you know?

d) Verify that the vector
\[
(2 - y(1), -1 - y(2), 0 - y(3), 5 - y(4))
\]
is orthogonal to \((1, 2, 3, 4)\) and \((1, 1, 1, 1)\), and explain why this is necessary.

e) Find the best-fit horizontal line \( y = D \) through these four points. Verify that \( D \) is the average of the \( y\)-values of the data points \( p_1, p_2, p_3, p_4 \).

**Solution.**

\[
\begin{array}{c}
\text{a),b)}
\end{array}
\]

\[
\begin{array}{c}
c) \text{The minimized quantity is}
\|b - b_v\|^2 = \sum_{i=1}^{4} (b_i - y(a_i))^2 = 16.
\end{array}
\]

d) We have
\[
\begin{pmatrix}
2 \\
-2 \\
-2 \\
2
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} = 0 \quad \begin{pmatrix}
2 \\
-2 \\
-2 \\
2
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} = 0.
\]

This is because \( b - b_v \) is orthogonal to the columns of the coefficient matrix of the system of equations
\[
\begin{align*}
C + D &= 2 \\
2C + D &= -1 \\
3C + D &= 0 \\
4C + D &= 5.
\end{align*}
\]

e) In this case we’re solving
\[
\begin{pmatrix}
1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} D = \begin{pmatrix}
1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
2 \\
-1 \\
0 \\
5
\end{pmatrix} \implies D = \frac{2 - 1 + 0 + 5}{4}.
\]
12. Consider the data points
\[ p_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad p_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \]

a) Find the best-fit parabola \( y = Cx^2 + Dx + E \) through these four points, and draw it on the grid below.

b) For each data point \( p_i = (a_i, b_i) \), draw the error bar from \( (a_i, y(a_i)) \) to \( (a_i, b_i) \).

c) What is the minimum value of \( \sum_{i=1}^{4} (b_i - y(a_i))^2 \)? How do you know?

Solution.

a),b)

c) The minimized quantity is
\[ \| b - b_v \|^2 = \sum_{i=1}^{4} (b_i - y(a_i))^2 = \frac{8}{5}. \]

13. Consider the following data points:
\[ p_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad p_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \quad p_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \]

a) Find the best-fit plane \( z = Cx + Dy + E \) through these four points.
b) Interpret the minimized quantity in the situation of this problem.

Solution.

a) The plane is $z = \frac{5}{4}x + \frac{5}{4}y + \frac{3}{4}$.

b) The minimized quantity is

$$\|b - b_V\|^2 = (z(1, 1) - 1)^2 + (z(1, -1) + 2)^2 + (z(-1, 1) - 3)^2 + (z(-1, -1) - 1)^2 = \frac{1}{4}.$$  

14. Consider the data points $p_1, \ldots, p_8$:

$$\left( \begin{array}{c}
1 \\
3 \\
1.5 \\
2.5 \\
0 \\
-1 \\
-1 \\
0 \\
2 \\
-1.5 \end{array} \right).$$

a) Find the best-fit ellipse

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

through these data points.

b) Interpret the minimized quantity in the situation of this problem.

[Hint: you can’t see it on the graph above, but you can see it on this demo.]

In this problem, I recommend using SymPy (in the Sage cell on the course webpage) or another computer algebra system to do the computations. To solve a normal equation $A^T Ax = A^T (1, 2, 3)$, you would use something like

$$(A.T*A).solve(A.T*Matrix([[1,2,3]]))$$

Remark: Carl Friedrich Gauss (1777–1865), arguably the greatest mathematician since antiquity, kept food on the table by doing astronomical calculations. He invented much of the linear algebra you are learning in order to compute the trajectories of celestial bodies. Essentially performing the calculations in this problem, he correctly predicted the (elliptical) orbit of the asteroid Ceres as it passed behind the sun in 1801.

Solution.

$$x^2 + 1.405y^2 + 0.395xy - 0.689x - 3.485y - 5.198 = 0$$
b) If \( z(x, y) = x^2 + By^2 + Cxy + Dx + Ey + F \) and \( p_i = (a_i, b_i) \), then the minimized quantity is

\[
\|b - \hat{b}\|^2 = \sum_{i=1}^{8} z(a_i, b_i)^2 \approx 6.60.
\]

15. Suppose that \( \hat{x} \) is a vector such that \( A\hat{x} = (1, 1, -1, -1) \). Explain why \( \hat{x} \) is not a least-squares solution of \( Ax = (1, 1, 1, 1) \).

Solution.
Because \( (1, 1, 1, 1) - A\hat{x} = (0, 0, 2, 2) \) is not orthogonal to \( (1, 1, -1, -1) \in \text{Col}(A) \).

16. Decide if each statement is true or false, and explain why.

a) A least-squares solution \( \hat{x} \) of \( Ax = b \) is a solution of \( A\hat{x} = b \) for \( V = \text{Col}(A) \).

b) Any solution of \( A^T A \hat{x} = A^T b \) is a least-squares solution of \( Ax = b \).

c) If \( A \) has full column rank, then \( Ax = b \) has exactly one least-squares solution for every \( b \).

d) If \( Ax = b \) has at least one least-squares solution for every \( b \), then \( A \) has full row rank.

Solution.

a) True.
b) True.
c) True.
d) False: \( Ax = b \) always has a least-squares solution.