

Math 218D-1: Homework #5

due Wednesday, February 14, at 11:59pm

- Give an example of a 3×3 matrix A such that $\text{Col}(A)$ contains $(1, 2, 3)$ and $(1, 0, -1)$, but $\text{Col}(A)$ is not all of \mathbf{R}^3 . What is the rank of A ?
 - Give an example of a 3×3 matrix A , with no zero entries, such that $\text{Col}(A)$ is the line through $(1, 1, 1)$. What is the rank of A ?
- For each matrix A , verify that $\text{rank}(A) = 1$, and find vectors u, v such that $A = uv^T$. (In general, a matrix has rank 1 if and only if it is equal to a column vector times a row vector.)

$$\text{a) } A = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 2 & 1 & -1 & 4 \\ -2 & -1 & 1 & -4 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$

- Find a basis for the null space of each matrix.

$$\text{a) } \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$
$$\text{c) } \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad \text{d) } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[Hint: You already did all of the work in HW4#14.]

- Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

$$\text{a) } \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \quad \text{b) } \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \quad \text{c) } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$
$$\text{d) } \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -3 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 6 \end{pmatrix} \right\} \quad \text{e) } \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Which sets do you know are linearly dependent without doing any work?

- For each set in Problem 4, find a basis for the span of the vectors.
 - For each set in Problem 4, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for a).
 - What is the dimension of each of these spans?

6. Consider the vectors

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$$

of Problem 4(a).

- a) Find two different ways to express $(5, 7, 9)$ as a linear combination of these vectors.
- b) How many ways can you express $(5, 7, 9)$ as a linear combination of the first two vectors?

7. Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V , and set

$$v_1 = w_2 + w_3 \quad v_2 = w_1 + w_3 \quad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V .

[Hint: You have to check spanning and linear independence by hand.]

8. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

9. Find bases for the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in \mathbf{R}^3 whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy -plane.

- 10.** Decide if each statement is true or false, and explain why.
- a) If v_1, v_2, \dots, v_n are linearly independent vectors, then $\text{Span}\{v_1, v_2, \dots, v_n\}$ has dimension n .
 - b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.
 - c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.
 - d) If v_3 is not a linear combination of v_1 and v_2 , then $\{v_1, v_2, v_3\}$ is linearly independent.
 - e) If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x .
 - f) The set $\{0\}$ is linearly independent.
 - g) If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.
 - h) The columns of any 4×5 matrix are linearly dependent.
 - i) If $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has only one solution, then the columns of A are linearly independent.
 - j) If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.

- 11.** Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that:
- (1) $\dim \text{Col}(A) + \dim \text{Nul}(A)$ is the number of columns of A .
 - (2) $\dim \text{Row}(A) + \dim \text{Nul}(A^T)$ is the number of rows of A .
 - (3) $\dim \text{Row}(A) = \dim \text{Col}(A)$.

[**Hint:** Augment with the identity matrix so you only have to do Gauss–Jordan elimination once. Feel free to use the Sage cell on the website!]

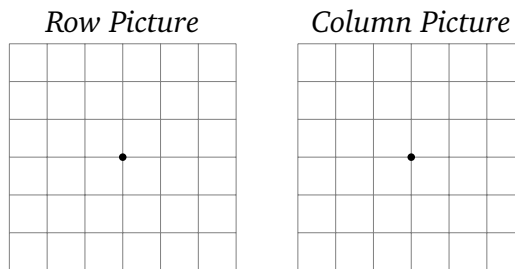
$$\begin{array}{l} \text{a) } \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \text{d) } \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad \text{e) } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

- 12.** Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.
- [**Hint:** No calculations are necessary.]

13. Find an example of a matrix with the required properties, or explain why no such matrix exists.
- The column space contains $(1, 2, 3)$ and $(4, 5, 6)$, and the row space contains $(1, 2)$ and $(2, 3)$.
 - The column space has basis $\{(1, 2, 3)\}$, and the null space has basis $\{(3, 2, 1)\}$.
 - The dimension of the null space is one greater than the dimension of the left null space.
 - A 3×5 matrix whose row space equals its null space.

14. Draw the four fundamental subspaces of the following matrices, in grids like below.

a) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$



15. For the following matrix A , find the pivot positions of A and of A^T . Do they have the same pivots? Do they have the same rank?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}$$

16. a) If $\text{Col}(B)$ is contained in $\text{Nul}(A)$, then $AB = \underline{\hspace{2cm}}$.
- b) Find a 2×2 matrix A such that $\text{Col}(A) = \text{Nul}(A)$. What is the rank of such a matrix? [**Hint:** use HW4#19.]
17. a) Show that $\text{rank}(AB) \leq \text{rank}(A)$. [**Hint:** Compare HW4#20.]
- b) Show that $\text{rank}(AB) \leq \text{rank}(B)$. [**Hint:** Take transposes in (a).]
18. Let A be a 3×3 matrix of rank 2. Explain why A^2 is not the zero matrix. [**Hint:** Compare Problem 16.]

19. This problem explains why we only consider *square* matrices when we discuss invertibility.
- a) Show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.
 - b) Show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.
- [Hint: Use Problem 17.]
20. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement “ A has full column rank”?
- a) $\text{Nul}(A) = \{0\}$
 - b) A has rank m
 - c) The columns of A are linearly independent
 - d) $\dim \text{Row}(A) = n$
 - e) The columns of A span \mathbf{R}^m
 - f) A^T has full column rank
21. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement “ A has full row rank”?
- a) $\text{Col}(A) = \mathbf{R}^m$
 - b) A has rank m
 - c) The columns of A are linearly independent
 - d) $\dim \text{Nul}(A) = n - m$
 - e) The rows of A span \mathbf{R}^n
 - f) A^T has full column rank