1. Draw the vectors
\[ v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad v + w, \quad v + 2w, \quad \text{and} \quad v - w \]
in a single xy-plane. What values of \( a \) and \( b \) make this equation true?
\[ av + bw = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \]

2. Consider the vectors
\[ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]
Draw the 16 linear combinations \( cv + dw \) (\( c,d = -1, 0, 1, 2 \)) as points in the xy-plane.

3. Certain vectors \( v, w \) in \( \mathbb{R}^2 \) are drawn below. Express each of \( b_1, b_2, b_3, b_4, b_5 \) as a linear combination of \( v, w \). Do not try to guess the coordinates of \( v \) and \( w \)! This is a question about the geometry of linear combinations.

4. Consider the vectors
\[ u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
Draw a picture of all of the linear combinations \( au + bv \) for real numbers \( a, b \) satisfying \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq 1 \). (This will be a shaded region in the xy-plane.)

5. Draw a picture of all vectors \( b \in \mathbb{R}^2 \) for which the equation
\[ \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} x = b \]
is consistent. [Hint: the answer is a span!]
6. For each matrix $A$ and vector $b$, and express the solution set in the form
\[ p + \text{Span}\{??\} \]
for some vector $p$. For instance,
\[ \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \text{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\} . \]

[Hint: You found the parametric vector form in HW3#12.]

a) \[ A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

b) \[ A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix} \]

c) \[ A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 10 \end{pmatrix} \]

d) \[ A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \]

7. For each matrix $A$ in Problem 6, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution?

[Hint: this problem requires no additional computation.]

8. When is the following system consistent?
\[
\begin{align*}
2x_1 + 2x_2 - x_3 &= b_1 \\
-4x_1 - 5x_2 + 5x_3 &= b_2 \\
6x_1 + x_2 + 12x_3 &= b_3
\end{align*}
\]

Your answer should be a single linear equation in $b_1, b_2, b_3$. [Hint: perform Gaussian elimination.]

Explain the relationship between this equation and
\[ \text{Span}\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\} . \]

9. Let $A$ be a $3 \times 4$ matrix whose columns span the plane $x + y + z = 0$.

a) Find a vector $b \in \mathbb{R}^3$ making the system $Ax = b$ consistent.

b) Find a vector $b \in \mathbb{R}^3$ making the system $Ax = b$ inconsistent.

[Hint: this problem requires no computations at all.]
10. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

11. Give geometric descriptions of the following spans (line, plane, ...).

   a) Span \( \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \)

   b) Span \( \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} \)

   c) Span \( \left\{ \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \)

   d) Span \( \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \\ \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\} \)

   e) Span \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \)

   [Hint: for d), compare Problem 8.]

12. a) List five nonzero vectors contained in Span \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \).

   b) Is \( \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \) contained in Span \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \)?

   If so, express \( \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \) as a linear combination of \( \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \).

   c) Show that \( \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \) is contained in Span \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \).

   d) Describe Span \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \) geometrically.

   e) Find a vector not contained in Span \( \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \).
13. Decide if each statement is true or false, and explain why.
   a) A vector \( b \) is a linear combination of the columns of \( A \) if and only if \( Ax = b \) has a solution.
   b) There is a matrix \( A \) such that \( Ax = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) has infinitely many solutions and \( Ax = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \) has exactly one solution.
   c) The zero vector is contained in every span.
   d) The matrix equation \( Ax = 0 \) can be consistent or inconsistent, depending on what \( A \) is.
   e) If the zero vector is a solution of a system of equations, then the system is homogeneous.
   f) If \( Ax = b \) has a unique solution, then \( A \) has a pivot in every column.
   g) If \( Ax = b \) is consistent, then the solution set of \( Ax = b \) is obtained by translating the solution set of \( Ax = 0 \).
   h) It is possible for \( Ax = b \) to have exactly 13 solutions.

14. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

   \[ a) \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b) \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \]
   \[ c) \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad d) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \]

   [Hint: You already did all of the work in Problem 7.]
15. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \):

\[ \text{Nul}(A) \quad \text{Col}(A) \]

b) \( A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \):

\[ \text{Nul}(A) \quad \text{Col}(A) \]

c) \( A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \):

\[ \text{Nul}(A) \quad \text{Col}(A) \]

16. Draw the solution set of the matrix equation

\[ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

Explain the relationship to your answer for Problem 15b.

17. Let \( A \) be a matrix such that \( \text{Nul}(A) = \text{Span}\{(1, 1, -1, -1), (1, -1, 1, -1)\} \). What is the rank of \( A \), and why?
18. Which of the following subsets of \( \mathbb{R}^3 \) are subspaces? If it is not a subspace, find a counterexample to one of the subspace properties. If it is, express it as the column space or null space of some matrix.
   a) The plane \( \{(x, y, x) : x, y \in \mathbb{R}\} \).
   b) The plane \( \{(x, y, 1) : x, y \in \mathbb{R}\} \).
   c) The set consisting of all vectors \((x, y, z)\) such that \( xy = 0 \).
   d) The set consisting of all vectors \((x, y, z)\) such that \( x \leq y \).
   e) The span of \((1, 2, 3)\) and \((2, 1, -3)\).
   f) The solution set of the system of equations \[
   \begin{align*}
   x + y + z &= 0 \\
   x - 2y - z &= 0.
   \end{align*}
   \]
   g) The solution set of the system of equations \[
   \begin{align*}
   x + y + z &= 0 \\
   x - 2y - z &= 1.
   \end{align*}
   \]

19. Find a nonzero \( 2 \times 2 \) matrix such that \( A^2 = 0 \).

20. a) Explain why \( \text{Col}(AB) \) is contained in \( \text{Col}(A) \).
   b) Give an example where \( \text{Col}(AB) \neq \text{Col}(A) \).
      [Hint: Take \( A = B \) to be the matrix from Problem 19.]

21. a) Explain why \( \text{Nul}(AB) \) contains \( \text{Nul}(B) \).
   b) Give an example where \( \text{Nul}(AB) \neq \text{Nul}(B) \).
      [Hint: Take \( A = B \) to be the matrix from Problem 19.]

22. Decide if each statement is true or false, and explain why.
   a) The column space of an \( m \times n \) matrix with \( m \) pivots is a subspace of \( \mathbb{R}^m \).
   b) The null space of an \( m \times n \) matrix with \( n \) pivots is equal to \( \mathbb{R}^n \).
   c) If \( \text{Col}(A) = \{0\} \), then \( A \) is the zero matrix.
   d) The column space of \( 2A \) equals the column space of \( A \).
   e) The null space of \( A + B \) contains the null space of \( A \).
   f) If \( U \) is an echelon form of \( A \), then \( \text{Nul}(U) = \text{Nul}(A) \).
   g) If \( U \) is an echelon form of \( A \), then \( \text{Col}(U) = \text{Col}(A) \).