Math 218D-1: Homework #10
Answer Key

1. For each $2 \times 2$ matrix $A$, i) compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$. Use this to ii) find all real eigenvalues, and iii) find a basis for each eigenspace, using HW9#14 when applicable. iv) Draw and label each eigenspace. v) Is the matrix diagonalizable (over the real numbers)?

\[ a) \begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix} \quad b) \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix} \quad c) \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad d) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad e) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

Solution.

a) $p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. The eigenvalues are 2 and 3, with eigenspaces $\text{Span}\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\}$ and $\text{Span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$, respectively.

This matrix is diagonalizable.

b) $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. The only eigenvalue is 2, and the 2-eigenspace is $\text{Span}\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\}$.

This matrix is not diagonalizable.

c) $p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. The only eigenvalue is 3, and the 3-eigenspace is $\mathbb{R}^2$.

This matrix is diagonalizable.

d) $p(\lambda) = \lambda^2 - 2\lambda + 2$. This polynomial has no real roots, so there are no real eigenvalues.
In particular, it is not diagonalizable (over the real numbers).

e) \( p(\lambda) = \lambda^2 - \lambda - 1. \) The eigenvalues are \( \frac{1}{2}(1 + \sqrt{5}) \) and \( \frac{1}{2}(1 - \sqrt{5}) \), with eigenspaces \( \text{Span}\{(1/\sqrt{5})\} \) and \( \text{Span}\{(1+\sqrt{5)/2}\} \), respectively.

This matrix is diagonalizable.

2. For each matrix \( A \), i) find all real eigenvalues of \( A \), and ii) find a basis for each eigenspace. iii) Is the matrix diagonalizable (over the real numbers)?

You will probably want to use a computer algebra system to find the roots of the characteristic polynomial. To do so in Sympy, you would type something like:

```
print(roots(-x**3 + 13/4*x + 3/2, multiple=True))
# [-1.5, -0.5, 2.0]
```

a) \[
\begin{pmatrix}
-1 & 7 & 5 \\
0 & 1 & -2 \\
0 & 1 & 4
\end{pmatrix}
\]

b) \[
\begin{pmatrix}
7 & 12 & 12 \\
-8 & -13 & -12 \\
4 & 6 & 5
\end{pmatrix}
\]

c) \[
\begin{pmatrix}
-14 & -7 & -12 \\
1 & 2 & 4
\end{pmatrix}
\]

Optional (if you want more practice):

d) \[
\begin{pmatrix}
-11 & -54 & 10 \\
-2 & -7 & 2 \\
-21 & -90 & 20
\end{pmatrix}
\]

e) \[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

f) \[
\begin{pmatrix}
13 & 18 & -18 \\
-12 & -17 & 18 \\
-4 & -6 & 7
\end{pmatrix}
\]

g) \[
\begin{pmatrix}
-10 & 28 & -18 & -76 \\
-1 & 9 & -6 & -2 \\
4 & -8 & 7 & 26 \\
0 & 2 & -2 & 4
\end{pmatrix}
\]

Solution.

a) \( p(\lambda) = -\lambda^3 + 4\lambda^2 - \lambda - 6 = -(\lambda - 3)(\lambda - 2)(\lambda + 1). \) The eigenvalues are 3, 2, and -1, with eigenspaces spanned by \((-1, -2, 2), (-3, -2, 1), \) and \((1, 0, 0), \) respectively. This matrix is diagonalizable.
b) \( p(\lambda) = -\lambda^3 - \lambda^2 + \lambda + 1 = -(\lambda - 1)(\lambda + 1)^2 \). The eigenvalues are 1 and -1, with eigenspaces
\[
\text{Span} \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \text{Span} \left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \right\},
\]
respectively. This matrix is diagonalizable.

c) \( p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 \). The only eigenvalue is 1, and the 1-eigenspace is
\[
\text{Span} \left\{ \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right\}.
\]
This matrix is not diagonalizable.

d) \( p(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda - 1)(\lambda - 2)(\lambda + 1) \). The eigenvalues are 1, 2, and -1, with eigenspaces spanned by \((7, -1, 3), (-6, 2, 3), (1, 0, 1)\), respectively. This matrix is diagonalizable.

e) \( p(\lambda) = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3 \). The only eigenvalue is 2, and the 2-eigenspace is \(\mathbb{R}^3\). The matrix is diagonalizable.

f) \( p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 \). The only eigenvalue is 1, and the 1-eigenspace is spanned by \((0, -3, 2)\). This matrix is not diagonalizable.

g) \( p(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24 = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) \). The eigenvalues are 1, 2, 3, and 4, with eigenspaces spanned by \((-16, 3, 6, 2), (-7, 1, 2, 1), (-10, 4, 5, 2), (-4, 2, 2, 1)\), respectively. This matrix diagonalizable.

3. Let \( V \) be the plane \( x + y + z = 0 \), and let \( R_V = I_3 - 2P_{V^1} \) be the reflection matrix over \( V \), as in HW9#6. Find an eigenbasis for \( R_V \) without doing any computations. Is \( R_V \) diagonalizable?

**Solution.**

In HW9#6 we computed
\[
R_V = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.
\]

Any vector on the plane \( V \) is fixed by \( R_V \), so \((-1, 1, 0)\) and \((-1, 0, 1)\) are 1-eigenvectors. Any vector on \( V^1 \) is negated by \( R_V \), so \((1, 1, 1)\) is a -1-eigenvector. Hence
\[
\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}
\]
is an eigenbasis, and \( R_V \) is diagonalizable.

4. The **Fibonacci numbers** are defined recursively as follows:
\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).
\]
The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ... In this problem, you will find a closed formula (as opposed to a recursive formula) for the $n$th Fibonacci number by solving a difference equation.

a) Let $v_n = \binom{F_{n+1}}{F_n}$, so $v_0 = \binom{0}{1}$, $v_1 = \binom{1}{1}$, etc. Find a state change matrix $A$ such that $v_{n+1} = Av_n$ for all $n \geq 0$.

b) Show that the eigenvalues of $A$ are $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$, with corresponding eigenvectors $w_1 = \binom{-1}{\lambda_2}$ and $w_2 = \binom{-1}{\lambda_1}$.

[Hint: Check that $Aw_1 = \lambda_1 w_1$ using the relations $\lambda_1 \lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$.]

c) Expand $v_0$ in this eigenbasis: that is, find $x_1, x_2$ such that $v_0 = x_1 w_1 + x_2 w_2$. (It helps to write $x_1, x_2$ in terms of $\lambda_1, \lambda_2$.)

d) Multiply $v_0 = x_1 w_1 + x_2 w_2$ by $A^n$ to show that $F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.

e) Use this formula to explain why $F_{n+1}/F_n$ approaches the golden ratio when $n$ is large.

Solution.

a) We have the system of equations

$$
\begin{align*}
F_{n+2} &= F_{n+1} + F_n \\
F_{n+1} &= F_n,
\end{align*}
$$

which translates into the matrix equation

$$
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}.
$$

b) We did this in Problem 1(e). We can also verify the eigenvectors directly using the relations $\lambda_1 \lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$:

$$
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
-1 \\
\lambda_2
\end{pmatrix} = \lambda_1
\begin{pmatrix}
-1 \\
\lambda_2
\end{pmatrix},
$$

$$
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
-1 \\
\lambda_1
\end{pmatrix} = \lambda_2
\begin{pmatrix}
-1 \\
\lambda_1
\end{pmatrix}.
$$

c) We can solve this by inspection:

$$
\begin{pmatrix}
\lambda_2 - \lambda_1 \\
0
\end{pmatrix} = \lambda_1
\begin{pmatrix}
-1 \\
\lambda_2
\end{pmatrix} - \lambda_2
\begin{pmatrix}
-1 \\
\lambda_1
\end{pmatrix} \implies v_0 = \frac{\lambda_1 w_1 - \lambda_2 w_2}{\lambda_2 - \lambda_1}.
$$

d) Multiplying the above equation by $A^n$ gives

$$
\begin{pmatrix}
F_{n+1} \\
F_n
\end{pmatrix} = v_n = \frac{\lambda_1^{n+1} w_1 - \lambda_2^{n+1} w_2}{\lambda_2 - \lambda_1}.
$$
The second coordinate of this is $F_n$; this works out to be

$$F_n = \frac{\lambda_2 \lambda_1^{n+1} - \lambda_1 \lambda_2^{n+1}}{\lambda_2 - \lambda_1} = \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1}.$$

**e)** Since $|\lambda_1| > |\lambda_2|$, when $n$ is large we have $\lambda_1^n - \lambda_2^n \approx \lambda_1^n$, so

$$\frac{F_{n+1}}{F_n} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} \approx \frac{\lambda_1^{n+1}}{\lambda_1^n} = \lambda_1.$$

5. Pretend that there are three Red Box kiosks in Durham. Let $x_t, y_t, z_t$ be the number of copies of Prognosis Negative at each of the three kiosks, respectively, on day $t$. Suppose in addition that a customer renting a movie from kiosk $i$ will return the movie the next day to kiosk $j$, with the following probabilities:

<table>
<thead>
<tr>
<th>Returning to kiosk</th>
<th>Renting from kiosk</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10% 30% 50%</td>
</tr>
<tr>
<td>2</td>
<td>20% 40% 30%</td>
</tr>
<tr>
<td>3</td>
<td>20% 40% 20%</td>
</tr>
</tbody>
</table>

For instance, a customer renting from kiosk 3 has a 50% probability of returning it to kiosk 1.

**a)** Let $v_t = (x_t, y_t, z_t)$. Find the state change matrix $A$ such that $v_{t+1} = Av_t$.

**b)** Diagonalize $A$. What are its eigenvalues? 

[Hint: $A$ is a stochastic matrix, so you know one eigenvalue by HW9#15(c).]

**c)** If you start with a total of 1000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk? Does it matter what the initial state is?

This is an example of a stochastic process, and is an important application of eigenvalues and eigenvectors.

**Solution.**

**a)** $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$

**b)** The eigenvalues of $A$ are $1, -2,$ and $.1$, with eigenvectors

$$w_1 = \begin{pmatrix} 1.4 \\ 1.2 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0.5 \\ -1.5 \\ 1 \end{pmatrix}.$$ 

**c)** If the starting state is $v_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$, then

$$A^n v_0 = x_1 w_1 + (-.2)^n x_2 w_2 + (.1)^n x_3 w_3.$$
Since \((-0.2)^n \to 0\) and \((0.1)^n \to 0\) as \(n \to \infty\), this approaches \(x_1 w_1\) as \(n \to \infty\). Hence the movies will be distributed in a 1.4 : 1.2 : 1 ratio; in percentages, this is approximately 38.9%, 33.3%, and 27.8%. Therefore, there will eventually be \(\approx 389\) movies in kiosk 1, \(\approx 333\) movies in kiosk 2, and \(\approx 278\) in kiosk 3. (This does not depend on the initial state \(v_0\).)

6. For each \(2 \times 2\) matrix \(A\) in Problem 1, if \(A\) is diagonalizable, find an invertible matrix \(C\) and a diagonal matrix \(D\) such that \(A = CDC^{-1}\).

**Solution.**

a) This matrix is equal to \(CDC^{-1}\) for
\[
C = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.
\]

b) This matrix is not diagonalizable.

c) This matrix is diagonal; it is equal to \(CDC^{-1}\) for \(D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\) and \(C = I_2\).

d) This matrix is not diagonalizable (over the real numbers).

e) This matrix is equal to \(CDC^{-1}\) for
\[
C = \begin{pmatrix} -1 \sqrt{5}/2 & -1 \sqrt{5}/2 \\ 1 - \sqrt{5}/2 & 1 + \sqrt{5}/2 \end{pmatrix}, \quad D = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{5} & 0 \\ 0 & 1 - \sqrt{5} \end{pmatrix}.
\]

7. For each matrix \(A\) in Problem 2, if \(A\) is diagonalizable, find an invertible matrix \(C\) and a diagonal matrix \(D\) such that \(A = CDC^{-1}\).

**Solution.**

a) This matrix is equal to \(CDC^{-1}\) for
\[
C = \begin{pmatrix} -1 & -3 & 1 \\ -2 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

b) This matrix is equal to \(CDC^{-1}\) for
\[
C = \begin{pmatrix} 2 & -3 & -3 \\ -2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

c) This matrix is not diagonalizable.

d) This matrix is equal to \(CDC^{-1}\) for
\[
C = \begin{pmatrix} 7 & -6 & 1 \\ -1 & 2 & 0 \\ 3 & 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

e) This matrix is diagonal; it is equal to \(CDC^{-1}\) for \(D = 2I_3\) and \(C = I_3\).

f) This matrix is not diagonalizable.
g) This matrix is equal to $CDC^{-1}$ for

$$C = \begin{pmatrix} -16 & -7 & -10 & -4 \\ 3 & 1 & 4 & 2 \\ 6 & 2 & 5 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

8. Consider the matrix

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

a) Find a diagonal matrix $D$ and an invertible matrix $C$ such that $A = CDC^{-1}$.

b) Find a different diagonal matrix $D'$ and a different invertible matrix $C'$ such that $A = C'D'C'^{-1}$.

[Hint: Try re-ordering the eigenvalues.]

Solution.

a) $C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b) $C' = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. Compute the matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(There is only one such matrix.)

Solution.

$$\begin{pmatrix} -2 & 2 & -1 \\ -2 & 2 & 0 \\ 2 & -2 & 3 \end{pmatrix}$$

10. Let $A$ and $B$ be $n \times n$ matrices, and let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$.

a) Suppose that each $v_i$ is an eigenvector of both $A$ and $B$. Show that $AB = BA$.

b) Suppose that each $v_i$ is an eigenvector of both $A$ and $B$ with the same eigenvalue. Show that $A = B$.

[Hint: Hint: use the matrix form of diagonalization.]

Solution.
a) We have $A = CDC^{-1}$ and $B = CD'C^{-1}$ for diagonal matrices $D, D'$ and the same matrix $C$ with columns $v_1, \ldots, v_n$. Noting that $DD' = D'D$ (multiplying diagonal matrices just multiplies the diagonal entries), we have

$$AB = (CDC^{-1})(CD'C^{-1}) = CDD'C^{-1} = CD'DC^{-1} = (CD'C^{-1})(CDC^{-1}) = BA.$$  

b) In this case we have $A = CDC^{-1}$ and $B = CD'C^{-1}$ where both $C$ and $D$ are the same.

11. Let $A$ be an $n \times n$ matrix, and let $C$ be an invertible $n \times n$ matrix. Prove that the characteristic polynomial of $CAC^{-1}$ equals the characteristic polynomial of $A$.

In particular, $A$ and $CAC^{-1}$ have the same eigenvalues, the same determinant, and the same trace. They are called similar matrices.

Solution.

First we note that $C(A - \lambda I_n)C^{-1} = (CA - \lambda C)C^{-1} = CAC^{-1} - \lambda I_n$, so

$$\det(CAC^{-1} - \lambda I_n) = \det(C(A - \lambda I_n)C^{-1}) = \det(C)\det(A - \lambda I_n)\det(C)^{-1} = \det(A - \lambda I_n).$$

12. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find a closed formula for $A^n$: that is, an expression of the form

$$A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},$$

where $a_{ij}(n)$ is a function of $n$.

Solution.

This is a diagonalization problem: $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

so

$$A^n = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 + 3^n & 1 - 3^n \\ 1 - 3^n & 1 + 3^n \end{pmatrix}.$$ 

13. A certain $2 \times 2$ matrix $A$ has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

a) Draw $Av$, $A^2v$, and $Aw$.

b) Compute the limit of $A^n v / \|A^n v\|$ as $n \to \infty$. 

Solution.

As $n \to \infty$, the vector $A^n v/\|A^n v\|$ tends to $\frac{1}{\sqrt{2}}(-1, -1)$.

14. A certain diagonalizable $2 \times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $w_1, w_2$ pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.
   a) Draw $C^{-1}v$ on the left.
   b) Draw $DC^{-1}v$ on the left.
   c) Draw $Av = CDC^{-1}v$ on the right.
   d) What happens to $A^n v$ as $n \to \infty$?
Solution.

As $n \to \infty$, the vector $A^n v$ approaches 0.