Math 218D-1: Homework #1

due Wednesday, September 6, at 11:59pm

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u + v + w$ and $u + 2v - w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to ____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Find two *different* triples (x, y, z) such that

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

4. Suppose that v and w are *unit vectors*: that is, $v \cdot v = 1$ and $w \cdot w = 1$. Compute the following dot products using the algebra of dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v + w) \cdot (v - w)$ **c)** $(v + 2w) \cdot (v - 2w)$.

5. Two vectors v and w are *orthogonal* if $v \cdot w = 0$, and they are *parallel* if one is a scalar multiple of the other. A *unit vector* is a vector v with $v \cdot v = 1$.

Decide if each statement is true or false, and explain why.

a) If $u = (1,1,1)$ is orthogonal to v and to w, then v is parallel to w.

b) If u is orthogonal to $v + w$ and to $v - w$, then u is orthogonal to v and w.

c) If u and v are orthogonal unit vectors then $(u - v) \cdot (u - v) = 2$.

d) If $u \cdot u + v \cdot v = (u + v) \cdot (u + v)$, then u and v are orthogonal.

6. Two vectors v and w are *orthogonal* if $v \cdot w = 0$. Find nonzero vectors v and w in $\mathbf{R}^3$ that are orthogonal to $(1,1,1)$ and to each other.

7. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why.

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 7 & 2 & 4 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 4 \\ -2 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 6 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 2 & 6 & -1 \end{pmatrix}$$

8. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

9. Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B - 3A$ **c)** AC **d)** B^2

e) $A + 2B$ **f)** $C - E$ **g)** EB **h)** D^2

10. Compute the product

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

in two ways:

a) using the column form, and

b) using the outer product form.

11. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

12. Consider the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Verify that $AC = BC$ and yet $A \neq B$.

13. Show that $(A+B)^2 \neq A^2 + 2AB + B^2$ when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

What is the correct formula?

$$(A+B)^2 = A^2 + B^2 + \underline{\hspace{2cm}}$$

[**Hint:** distribute the product $(A+B)(A+B)$.]

14. In the following, find the 2×2 matrix A that acts in the specified manner.

a) $A\binom{x}{y} = \binom{x}{y}$: the identity matrix does not change the vector.

b) $A\binom{x}{y} = \binom{y}{x}$: this is a flip over the line $y = x$.

c) $A\binom{x}{y} = \binom{y}{-x}$: this rotates vectors clockwise by 90°.

d) $A\binom{x}{y} = -\binom{x}{y}$: this rotates vectors by 180°.

e) $A\binom{x}{y} = \binom{0}{y}$: this projects onto the y-axis.

f) $A\binom{x}{y} = \binom{x}{0}$: this projects onto the x-axis.

g) $A\binom{x}{y} = \binom{x}{y-2x}$: this performs the row operation $R_2 -= 2R_1$.

[**Hint:** compute $A\binom{1}{0}$ and $A\binom{0}{1}$.]

15. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by the diagonal matrix D on the right or the left.

16. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \qquad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \qquad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

What is A?

17. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \\ 9 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \\ 3 \end{pmatrix}.$$

Compute Ax, where x is the vector $(3, -2, -2)$.
[**Hint:** express $(3, -2, -2)$ as a linear combination of $(1, 0, 2)$ and $(0, 1, 4)$.]

18. For the following matrices A and B, compute $AB, A^T, B^T, B^T A^T$, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute $A^T B^T$?

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.$$

19. Recall that a matrix A is *symmetric* if $A^T = A$. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is any matrix, then $A^T A$ is symmetric.