1. Consider the vectors

\[ u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}. \]

a) Compute \( u + v + w \) and \( u + 2v - w \).

b) Find numbers \( x \) and \( y \) such that \( w = xu + yv \).

c) Explain why every linear combination of \( u, v, w \) is also a linear combination of \( u \) and \( v \) only.

d) The sum of the coordinates of any linear combination of \( u, v, w \) is equal to _____?

e) Find a vector in \( \mathbb{R}^3 \) that is not a linear combination of \( u, v, w \).

2. Find two different triples \((x, y, z)\) such that

\[ x \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \]

How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector \( \frac{1}{2}v \) is a linear combination of \( v \) and \( w \).

b) \[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

c) If \( v, w \) are two vectors in \( \mathbb{R}^2 \), then any other vector \( b \) in \( \mathbb{R}^2 \) is a linear combination of \( v \) and \( w \).

4. Suppose that \( v \) and \( w \) are unit vectors; that is, \( v \cdot v = 1 \) and \( w \cdot w = 1 \). Compute the following dot products using the algebra of dot products (your answers will be actual numbers):

a) \( v \cdot (-v) \) \quad b) \((v + w) \cdot (v - w)\) \quad c) \((v + 2w) \cdot (v - 2w)\).
5. Two vectors $v$ and $w$ are orthogonal if $v \cdot w = 0$, and they are parallel if one is a scalar multiple of the other. A unit vector is a vector $v$ with $v \cdot v = 1$.

Decide if each statement is true or false, and explain why.

a) If $u = (1, 1, 1)$ is orthogonal to $v$ and to $w$, then $v$ is parallel to $w$.

b) If $u$ is orthogonal to $v + w$ and to $v - w$, then $u$ is orthogonal to $v$ and $w$.

c) If $u$ and $v$ are orthogonal unit vectors then $(u - v) \cdot (u - v) = 2$.

d) If $u \cdot u + v \cdot v = (u + v) \cdot (u + v)$, then $u$ and $v$ are orthogonal.

6. Two vectors $v$ and $w$ are orthogonal if $v \cdot w = 0$. Find nonzero vectors $v$ and $w$ in $\mathbb{R}^3$ that are orthogonal to $(1, 1, 1)$ and to each other.

7. Compute the following matrix-vector products using both the by-row and by-column methods. If the product is not defined, explain why.

\[
\begin{pmatrix}
2 & 1 \\
5 & -3 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
1 & -2 \\
0 & -1 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
-2
\end{pmatrix}
\begin{pmatrix}
7 & 2 & 4 \\
3 & -3 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
7 & 4 \\
-2 & 2 \\
4 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
-2
\end{pmatrix}
\begin{pmatrix}
1 & -2 \\
0 & -1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
5 \\
5 \\
-1
\end{pmatrix}
\begin{pmatrix}
5 \\
2 \\
6 \\
-1
\end{pmatrix}
\]

8. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector $w$ in $\mathbb{R}^2$ such that

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

9. Consider the matrices

\[A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.\]

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$  b) $B - 3A$  c) $AC$  d) $B^2$

e) $A + 2B$  f) $C - E$  g) $EB$  h) $D^2$
10. Compute the product
\[
\begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & -1 \\
4 & -1 & 2
\end{pmatrix}
\]

in two ways:
\(a)\) using the column form, and
\(b)\) using the outer product form.

11. Consider the matrices
\[
A = \begin{pmatrix}
1 & 2 \\
-2 & 1
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 1 \\
-1 & h
\end{pmatrix}.
\]
What value(s) of \(h\), if any, will make \(AB = BA\)?

12. Consider the matrices
\[
A = \begin{pmatrix}
1 & -3 \\
2 & 5
\end{pmatrix} \quad B = \begin{pmatrix}
-4 & -8 \\
5 & 8
\end{pmatrix} \quad C = \begin{pmatrix}
-1 & -1 \\
1 & 1
\end{pmatrix}.
\]
Verify that \(AC = BC\) and yet \(A \neq B\).

13. Show that \((A + B)^2 \neq A^2 + 2AB + B^2\) when
\[
A = \begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}.
\]
What is the correct formula?
\[
(A + B)^2 = A^2 + B^2 + ______
\]
[\text{Hint: distribute the product } (A + B)(A + B).]\]

14. In the following, find the \(2 \times 2\) matrix \(A\) that acts in the specified manner.
\(a)\) \(A_{x}^{y} = (y)\): the identity matrix does not change the vector.
\(b)\) \(A_{x}^{y} = (y)\): this is a flip over the line \(y = x\).
\(c)\) \(A_{x}^{y} = (y)\): this rotates vectors clockwise by \(90^\circ\).
\(d)\) \(A_{x}^{y} = -(x)\): this rotates vectors by \(180^\circ\).
\(e)\) \(A_{x}^{y} = (y)\): this projects onto the \(y\)-axis.
\(f)\) \(A_{x}^{y} = (x)\): this projects onto the \(x\)-axis.
\(g)\) \(A_{x}^{y} = (\frac{x}{y} - 2x)\): this performs the row operation \(R_2 = -2R_1\).
[\text{Hint: compute } A_{x}^{y} \text{ and } A_{x}^{y}.]
15. Consider the matrices
\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.
\]
Compute \(AD\) and \(DA\). Explain how the columns or rows of \(A\) change when \(A\) is multiplied by the diagonal matrix \(D\) on the right or the left.

16. Let \(A\) be a \(4 \times 3\) matrix satisfying
\[
Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \quad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \quad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.
\]
What is \(A\)?

17. Suppose that \(A\) is a \(4 \times 3\) matrix such that
\[
A \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \\ 9 \end{pmatrix} \quad A \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \\ 3 \end{pmatrix}.
\]
Compute \(Ax\), where \(x\) is the vector \((3, -2, -2)\).
[Hint: express \((3, -2, -2)\) as a linear combination of \((1, 0, 2)\) and \((0, 1, 4)\).]

18. For the following matrices \(A\) and \(B\), compute \(AB, A^T, B^T, B^TA^T,\) and \((AB)^T\). Which of these matrices are equal and why? Why can't you compute \(A^TB^T\)?
\[
A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.
\]

19. Recall that a matrix \(A\) is symmetric if \(A^T = A\). Decide if each statement is true or false, and explain why.
   a) If \(A\) and \(B\) are symmetric of the same size, then \(AB\) is symmetric.
   b) If \(A\) is symmetric, then \(A^3\) is symmetric.
   c) If \(A\) is any matrix, then \(A^TA\) is symmetric.