Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a simple calculator for doing arithmetic, but you should not need one. You may bring a 3 × 5-inch note card covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!
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Problem 1. [20 points]

Consider the following matrix $A$ and its reduced row echelon form:

$$A = \begin{pmatrix} 3 & -1 & -2 & -1 \\ 5 & 1 & -6 & -7 \\ 1 & -3 & 2 & 5 \\ -2 & 2 & 0 & -2 \end{pmatrix} \quad \text{RREF} \quad \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Let $V$ be the subspace Nul($A$).

a) $V$ is a (fill in one bubble)
   - point
   - line
   - plane
   - space
   in $\mathbb{R}^n$.

b) Find a basis for $V$.
   $\left\{ \right\}$
   $\left\{ \right\}$
   $\left\{ \right\}$
   $\left\{ \right\}$

c) Find a basis for $V^\perp$.
   $\left\{ \right\}$
   $\left\{ \right\}$
   $\left\{ \right\}$
   $\left\{ \right\}$

d) Compute the orthogonal projection $b_V$ for $b = (0, 1, -1, 1)$.
   $b_V =$
   $\left( \right)$
   $\left( \right)$
   $\left( \right)$
   $\left( \right)$
[Scratch work for Problem 1]
(Problem 1, continued)

e) Compute the projection matrix $P_V$.

$$P_V = \begin{pmatrix} \end{pmatrix}$$

f) Compute the projection matrix $P_{V\perp}$.

$$P_{V\perp} = \begin{pmatrix} \end{pmatrix}$$
[Scratch work for Problem 1]
Problem 2. [15 points]

In this problem we will find the best fit plane

\[ z = B + Cx + Dy \]

through the data points

\[
\begin{pmatrix}
 x \\
 y \\
 z
\end{pmatrix} = \begin{pmatrix}
 3 \\
 1 \\
 1
\end{pmatrix}, \quad \begin{pmatrix}
 3 \\
 -1 \\
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\end{pmatrix}, \quad \begin{pmatrix}
 1 \\
 -1 \\
 1
\end{pmatrix}, \quad \begin{pmatrix}
 1 \\
 -3 \\
 3
\end{pmatrix}.
\]

a) Find the matrix \( A \) and the vector \( b \) such that the coefficients \((B, C, D)\) of the best-fit plane is the least-squares solution of the matrix equation \( A\begin{pmatrix} B \\ C \\ D \end{pmatrix} = b \).

\[
A = \begin{pmatrix}
 & & \\
 & & \\
 & & \\
\end{pmatrix} \quad b = \begin{pmatrix}
 & \\
 & \\
 & \\
\end{pmatrix}
\]

b) Compute the QR decomposition of \( A \).

\[
Q = \begin{pmatrix}
 & & \\
 & & \\
 & & \\
\end{pmatrix} \quad R = \begin{pmatrix}
 & & \\
 & & \\
 & & \\
\end{pmatrix}
\]
[Scratch work for Problem 2]
(Problem 2, continued)

c) Find the best-fit plane, using your answer to b) or otherwise.

\[ z = \square + \square x + \square y \]

d) Find the orthogonal projection of the vector \( b \) onto \( V = \text{Col}(A) \).

\[ b_V = \begin{pmatrix} \square \\ \square \end{pmatrix} \]
[Scratch work for Problem 2]
Problem 3. [15 points]

Consider the matrix

\[ A = \begin{pmatrix} 1 & -4 & 8 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{pmatrix}. \]

a) Compute the characteristic polynomial of \( A \).

\[ p(\lambda) = \]

b) Compute a basis for the 1-eigenspace of \( A \).

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[Scratch work for Problem 3]
Problem 4. [10 points]

A subspace $V$ and a vector $b$ are drawn below.

a) Draw and label the vectors $b_V$ and $b_{V\perp}$.

b) Draw and label any eigenvector of the projection matrix $P_V$.

A different subspace $W$ is drawn below. Grid marks are one unit apart.

c) Compute the projection matrix $P_W$.

$$P_W = \begin{pmatrix} \ \ \ \ \\
\end{pmatrix}$$
[Scratch work for Problem 4]
Problem 5. [20 points]

Short-answer questions: no justification is necessary.

a) \[
\begin{pmatrix}
5 & 6 \\
-1 & 0
\end{pmatrix}^{15} \begin{pmatrix}
2 \\
-1
\end{pmatrix} =
\]

b) Suppose that \( V \) is a subspace of \( \mathbb{R}^n \). Which of the following statements are equivalent to “\( \dim(V) = n \)”? Fill in the bubbles of all that apply.

- \( \det(P_V) \neq 0 \)
- \( V = \mathbb{R}^n \)
- 1 is an eigenvalue of \( P_V \)
- \( P_{V^\perp} = 0 \)
- \( \text{rank}(P_V) = n \)
- \( b_V = b \) for every \( b \in V \)

\[ \]

c) Compute the area of the triangle in the picture. (Grid marks are one unit apart.)

\[ \text{area} = \]

\[ \]

d) Find a basis for the orthogonal complement of

\[ V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} \right\}. \]

\[ \]

e) Which one of the following statements is correct? Fill in the appropriate bubble.

- An eigenvector of \( A \) is a vector \( v \) such that \( Av = \lambda v \) for a nonzero scalar \( \lambda \).
- An eigenvector of \( A \) is a nonzero vector \( v \) such that \( Av = \lambda v \) for a scalar \( \lambda \).
- An eigenvector of \( A \) is a nonzero scalar \( \lambda \) such that \( Av = \lambda v \) for some vector \( v \).
- An eigenvector of \( A \) is a vector \( v \) such that \( Av = \lambda v \) for a scalar \( \lambda \).
[Scratch work for Problem 5]
Problem 6.  [20 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead. You need not justify your answers.

a) A matrix $Q$ with orthonormal columns such that $P_V = QQ^T$, where $V$ is the plane $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right\}.$

b) An invertible matrix $A$ with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

c) A non-invertible matrix $A$ such that $A^T A$ is invertible.

d) A $2 \times 2$ projection matrix of rank 1.
[Scratch work for Problem 6]