MATH 218D-1
MIDTERM EXAMINATION 2

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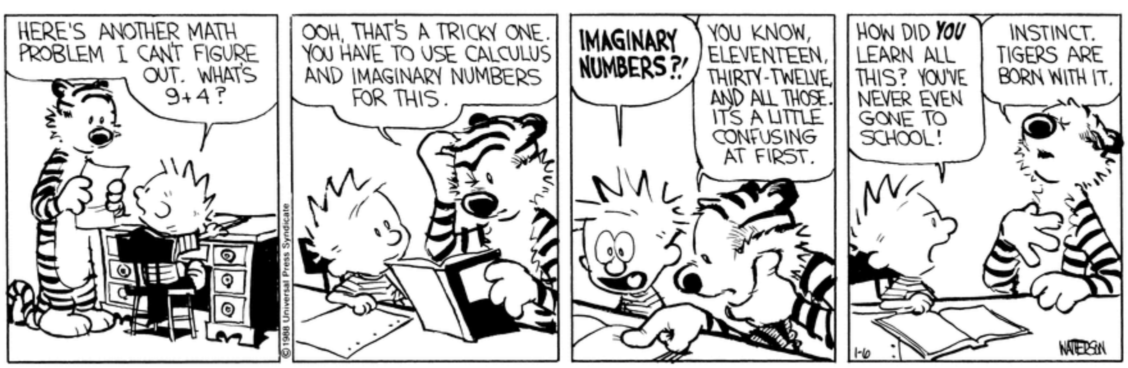
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1. [20 points]

Consider the following matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 3 & -1 & -2 & -1 \\ 5 & 1 & -6 & -7 \\ 1 & -3 & 2 & 5 \\ -2 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let V be the subspace Nul(A).

a) V is a (fill in one bubble)

- ○ point
- ○ line
- ● plane
- ○ space

in $\mathbf{R}^{\boxed{4}}$.

b) Find a basis for V.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

c) Find a basis for $V^{\perp}$.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -2 \end{pmatrix} \right\}$$

d) Compute the orthogonal projection b_V for $b = (0, 1, -1, 1)$.

$$b_V = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

e) Compute the projection matrix P_V.

$$P_V = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

f) Compute the projection matrix $P_{V^{\perp}}$.

$$P_{V^{\perp}} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

Problem 2. [15 points]

In this problem we will find the best fit plane

$$z = B + Cx + Dy$$

through the data points

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}.$$

a) Find the matrix A and the vector b such that the coefficients (B, C, D) of the best-fit plane is the least-squares solution of the matrix equation $A\begin{pmatrix} B \\ C \\ D \end{pmatrix} = b$.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

b) Compute the QR decomposition of A.

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

c) Find the best-fit plane, using your answer to **b)** or otherwise.

$$z = \boxed{-2} + \boxed{1}\,x + \boxed{-1}\,y$$

d) Find the orthogonal projection of the vector b onto $V = \mathrm{Col}(A)$.

$$b_V = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}$$

Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -4 & 8 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{pmatrix}.$$

a) Compute the characteristic polynomial of A.

$$p(\lambda) = -\lambda^3 + \lambda^2 + \lambda - 1$$

b) Compute a basis for the 1-eigenspace of A.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

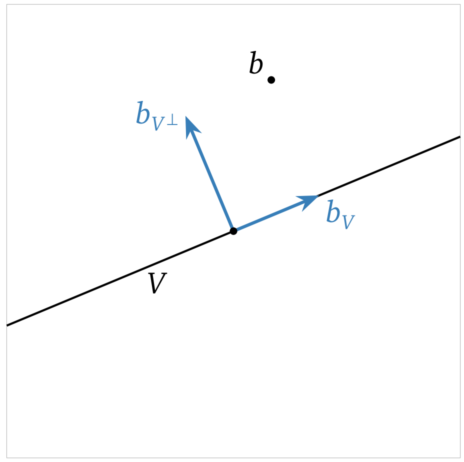
c) Is 2 an eigenvalue of A? Why or why not?

No, because $p(2) = -3 \neq 0$.

Problem 4.

[10 points]

A subspace V and a vector b are drawn below.

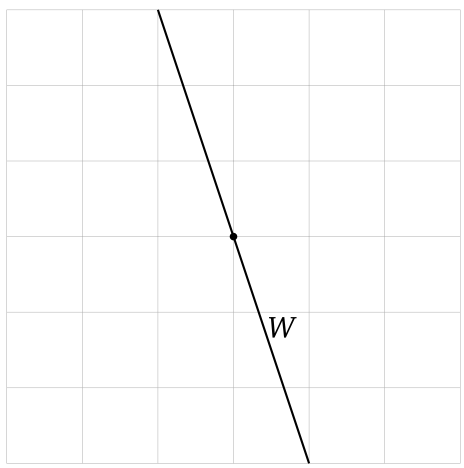


a) Draw and label the vectors b_V and $b_{V^\perp}$.

b) Draw and label any eigenvector of the projection matrix P_V.

Any nonzero vector on V or on $V^\perp$ works.

A different subspace W is drawn below. Grid marks are one unit apart.



c) Compute the projection matrix P_W.

$$P_W = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$$

Problem 5. [20 points]

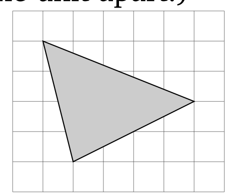
Short-answer questions: no justification is necessary.

a) $\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}^{15} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2^{15}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b) Suppose that V is a subspace of $\mathbf{R}^n$. Which of the following statements are *equivalent* to "$\dim(V) = n$"? Fill in the bubbles of all that apply.

- ● $\det(P_V) \neq 0$
- ● $V = \mathbf{R}^n$
- ○ 1 is an eigenvalue of P_V
- ● $P_{V^\perp} = 0$
- ● $\text{rank}(P_V) = n$
- ○ $b_V = b$ for every $b \in V$

c) Compute the area of the triangle in the picture. (Grid marks are one unit apart.)



area = 9

d) Find a basis for the orthogonal complement of

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} \right\}.$$

$$\left\{ \begin{pmatrix} -11 \\ 9 \\ -8 \end{pmatrix} \right\}$$

e) Which **one** of the following statements is correct? Fill in the appropriate bubble.

- ○ An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
- ● An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
- ○ An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
- ○ An eigenvector of A is a vector v such that $Av = \lambda v$ for a scalar λ.

Problem 6. [20 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A matrix Q *with orthonormal columns* such that $P_V = QQ^T$, where V is the plane

$$V = \text{Span}\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}4\\4\\-1\end{pmatrix}\right\}.$$

We need to find an orthonormal basis for V. Running Gram–Schmidt and normalizing, we can use

$$Q = \begin{pmatrix}1/\sqrt{14} & 3/\sqrt{19}\\ 3/\sqrt{14} & 1/\sqrt{19}\\ 2/\sqrt{14} & -3/\sqrt{19}\end{pmatrix}.$$

b) An invertible matrix A with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

Impossible: zero is an eigenvalue of A because $p(0) = 0$.

c) A non-invertible matrix A such that $A^T A$ is invertible.

Any tall matrix with full column rank will work. For instance,

$$A = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}.$$

d) A 2×2 projection matrix of rank 1.

Use the formula for projection onto any line in $\mathbf{R}^2$. For instance, the matrix for projection onto the x-axis is

$$P_V = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$