

**MATH 218D-1**  
**PRACTICE MIDTERM EXAMINATION 2**

<b>Name</b>		<b>Duke NetID</b>	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a **3 × 5-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

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## Problem 1.

[20 points]

Consider the subspace

a) Find an orthogonal basis for

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\left\{ \quad \quad \quad \right\}$$

[Scratch work for Problem 1]

**(Problem 1, continued)**

Now we change subspaces to avoid carry-through error. Consider the subspace

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\},$$

and note that the spanning vectors are *orthogonal*.

**b)** Compute the orthogonal projection of  $b = (-7, 4, -4)$  onto  $V$ .

$$b_V = \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix}$$

**c)** Compute the matrix  $P_V$  for projection onto  $V$ .

$$P_V = \begin{pmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{pmatrix}$$

**d)** Compute the matrix  $P_{V^\perp}$  for projection onto  $V^\perp$ .

$$P_{V^\perp} = \begin{pmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{pmatrix}$$

**e)** Find a basis of  $\text{Nul}(P_V)$ .

$$\left\{ \begin{pmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{pmatrix} \right\}$$

[Scratch work for Problem 1]

## Problem 2.

[15 points]

In this problem we will consider the best-fit line  $y = Cx + D$  through the data points

$$\begin{pmatrix} 1 \\ b_1 \end{pmatrix}, \begin{pmatrix} 2 \\ b_2 \end{pmatrix}, \begin{pmatrix} 3 \\ b_3 \end{pmatrix}, \begin{pmatrix} 4 \\ b_4 \end{pmatrix}.$$

a) The line  $y = Cx + D$  passes through all four points if and only if the matrix equation

$$\begin{pmatrix} \phantom{1} \\ \phantom{2} \\ \phantom{3} \\ \phantom{4} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

is satisfied (fill in the blank).

Let  $A$  be the coefficient matrix in the previous problem. In the QR decomposition of  $A$ , the matrix  $Q$  is

$$Q = \begin{pmatrix} 1/\sqrt{30} & 2/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{6} \\ 3/\sqrt{30} & 0 \\ 4/\sqrt{30} & -1/\sqrt{6} \end{pmatrix}.$$

b) Explain why  $R = Q^T A$ , and compute  $R$ .

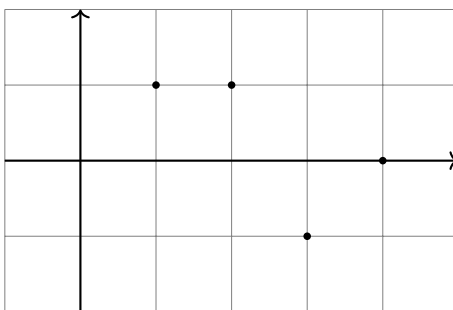
$$R = \begin{pmatrix} \phantom{1} \\ \phantom{2} \\ \phantom{3} \\ \phantom{4} \end{pmatrix}$$

c) Use the QR decomposition to find the best-fit line through the data points

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$$y = \boxed{\phantom{000}}x + \boxed{\phantom{000}}$$

d) Graph the line you found in c) below. Explain which quantity was minimized in terms of the graph.



[Scratch work for Problem 2]



### Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ -1 & -1 & 1 & 1 \\ 3 & 3 & -1 & -5 \end{pmatrix}.$$

a) Compute bases of all four fundamental subspaces of  $A$ .

$$\begin{array}{l} \text{Nul}(A): \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\} \\ \text{Row}(A): \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\} \end{array} \quad \begin{array}{l} \text{Col}(A): \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\} \\ \text{Nul}(A^T): \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\} \end{array}$$

b) Compute the orthogonal decomposition of  $(0, 3, 3)$  with respect to  $V = \text{Col}(A)$ .

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} =$$

c) Compute the distance from  $(0, 3, 3)$  to  $\text{Col}(A)$ .

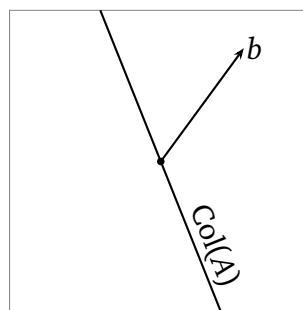
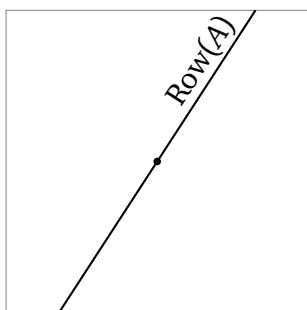


[Scratch work for Problem 3]

### Problem 4.

[10 points]

For a certain  $2 \times 2$  matrix  $A$ , the row space of  $A$  is drawn in the picture on the left, and the column space is drawn in the picture on the right.



a)  $\text{rank}(A) =$   .

b) Draw  $\text{Nul}(A)$  in the picture on the left.

c) Draw  $\text{Nul}(A^T)$  in the picture on the right.

d) If  $V = \text{Col}(A)$  and  $b$  is the vector in the picture on the right, draw and label the vectors  $b_V$  and  $b_{V^\perp}$ .

[Scratch work for Problem 4]

## Problem 5.

[20 points]

a) Find a matrix whose null space is  $\text{Span}\{(1, 1, 1)\}$ .

b) For which value(s) of  $k$ , if any, do the following vectors *not* form a basis of  $\mathbf{R}^4$ ?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ k \end{pmatrix} \right\}$$

c) Which of the following properties of a matrix are not changed by row operations?  
(Fill in the bubbles of all that apply)

- |  |  |
|--|--|
| <input type="radio"/> The rank         | <input type="radio"/> The left null space          |
| <input type="radio"/> The column space | <input type="radio"/> The determinant              |
| <input type="radio"/> The null space   | <input type="radio"/> The reduced row echelon form |
| <input type="radio"/> The row space    |  |

d) If  $\det(A) = 2$  and  $\det(B) = 3$ , compute the following determinants:

$$\det(A^2) = \boxed{\phantom{000}} \quad \det(AB^T) = \boxed{\phantom{000}} \quad \det(BA^k B^{-1}) = \boxed{\phantom{000}}$$

(Here  $A$  and  $B$  are square matrices of the same size and  $k$  is a whole number.)

[Scratch work for Problem 5]

## Problem 6.

[20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

a) A  $3 \times 2$  matrix  $A$  such that  $Ax = (1, 2, 3)$  has more than one least-squares solution.

b) A matrix  $A$  in RREF satisfying  $\dim \text{Row}(A) = 2$  and  $\dim \text{Nul}(A) = 3$ .

c) A matrix  $Q$  with orthonormal columns, such that  $\det(QQ^T) = 0$ .

d) A matrix  $A$  whose column space  $V = \text{Col}(A)$  is a plane in  $\mathbf{R}^3$ , such that  $\text{rank}(P_V) = 1$ .

[Scratch work for Problem 6]