MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

Consider the subspace

a) Find an orthogonal basis for

$$\text{Span}\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},\begin{pmatrix}3\\3\\-1\\-1\end{pmatrix},\begin{pmatrix}2\\2\\0\\0\end{pmatrix}\right\}.$$

$$\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$$

Now we change subspaces to avoid carry-through error. Consider the subspace

$$V=\text{Span}\left\{\begin{pmatrix}1\\2\\-2\end{pmatrix},\begin{pmatrix}-2\\2\\1\end{pmatrix}\right\},$$

and note that the spanning vectors are *orthogonal*.

b) Compute the orthogonal projection of $b=(-7,4,-4)$ onto V.

$$b_V=\begin{pmatrix}-3\\6\\0\end{pmatrix}$$

c) Compute the matrix P_V for projection onto V.

$$P_V=\frac{1}{9}\begin{pmatrix}5&-2&-4\\-2&8&-2\\-4&-2&5\end{pmatrix}$$

d) Compute the matrix $P_{V^\perp}$ for projection onto $V^\perp$.

$$P_{V^\perp}=\frac{1}{9}\begin{pmatrix}4&2&4\\2&1&2\\4&2&4\end{pmatrix}$$

e) Find a basis of $\text{Nul}(P_V)$.

$$\left\{\begin{pmatrix}4\\2\\4\end{pmatrix}\right\}$$

Problem 2. [15 points]

In this problem we will consider the best-fit line $y = Cx + D$ through the data points

$$\begin{pmatrix}1\\b_1\end{pmatrix}, \quad \begin{pmatrix}2\\b_2\end{pmatrix}, \quad \begin{pmatrix}3\\b_3\end{pmatrix}, \quad \begin{pmatrix}4\\b_4\end{pmatrix}.$$

a) The line $y = Cx + D$ passes through all four points if and only if the matrix equation

$$\begin{pmatrix}1 & 1\\2 & 1\\3 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$$

is satisfied (fill in the blank).

Let A be the coefficient matrix in the previous problem. In the QR decomposition of A, the matrix Q is

$$Q = \begin{pmatrix}1/\sqrt{30} & 2/\sqrt{6}\\2/\sqrt{30} & 1/\sqrt{6}\\3/\sqrt{30} & 0\\4/\sqrt{30} & -1/\sqrt{6}\end{pmatrix}.$$

b) Explain why $R = Q^T A$, and compute R.

Since $Q^T Q = I_2$, multiplying both sides of $A = QR$ by Q^T gives $Q^T A = R$.

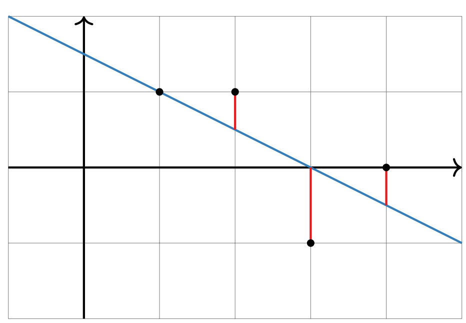
$$R = \begin{pmatrix}\sqrt{30} & \sqrt{30}/3\\0 & \sqrt{6}/3\end{pmatrix}$$

c) Use the QR decomposition to find the best-fit line through the data points

$$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}3\\-1\end{pmatrix}, \quad \begin{pmatrix}4\\0\end{pmatrix}.$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

d) Graph the line you found in **c)** below. Explain which quantity was minimized in terms of the graph.



The sum of the squares of the lengths of the red lines is minimized.

Problem 3. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ -1 & -1 & 1 & 1 \\ 3 & 3 & -1 & -5 \end{pmatrix}.$$

a) Compute bases of all four fundamental subspaces of A.

$$\mathrm{Nul}(A): \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \mathrm{Col}(A): \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\mathrm{Row}(A): \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} \qquad \mathrm{Nul}(A^T): \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

b) Compute the orthogonal decomposition of $(0, 3, 3)$ with respect to $V = \mathrm{Col}(A)$.

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

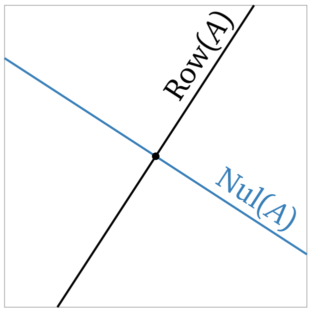
c) Compute the distance from $(0, 3, 3)$ to $\mathrm{Col}(A)$.

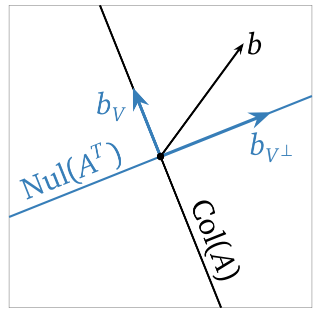
$$\sqrt{6}$$

Problem 4.

[10 points]

For a certain 2×2 matrix A, the row space of A is drawn in the picture on the left, and the column space is drawn in the picture on the right.





a) $\text{rank}(A) = 1$.

b) Draw $\text{Nul}(A)$ in the picture on the left.

c) Draw $\text{Nul}(A^T)$ in the picture on the right.

d) If $V = \text{Col}(A)$ and b is the vector in the picture on the right, draw and label the vectors b_V and $b_{V^\perp}$.

Problem 5. [20 points]

a) Find a matrix whose null space is $\mathrm{Span}\{(1,1,1)\}$.

The orthogonal complement of $\mathrm{Span}\{(1,1,1)\}$ is $\mathrm{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$, which has basis $\{(-1,1,0),(-1,0,1)\}$. Taking orthogonal complements again, we have

$$\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} = \mathrm{Nul}\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}.$$

b) For which value(s) of k, if any, do the following vectors *not* form a basis of $\mathbf{R}^4$?

$$\left\{\begin{pmatrix}1\\0\\0\\-6\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-8\end{pmatrix},\begin{pmatrix}1\\2\\3\\k\end{pmatrix}\right\}$$

Expanding cofactors along the first row, we compute

$$\det\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\\ -6 & -1 & -8 & k\end{pmatrix} = \det\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\\ -1 & -8 & k\end{pmatrix} + 6\det\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 2\\ 0 & 1 & 3\end{pmatrix}$$
$$= k + 32.$$

Hence the columns are not linearly independent exactly when $k = -32$.

c) Which of the following properties of a matrix are not changed by row operations? (Fill in the bubbles of all that apply)

- ● The rank
- ○ The column space
- ● The null space
- ● The row space
- ○ The left null space
- ○ The determinant
- ● The reduced row echelon form

d) If $\det(A) = 2$ and $\det(B) = 3$, compute the following determinants:

$$\det(A^2) = 4 \qquad \det(AB^T) = 6 \qquad \det(BA^kB^{-1}) = 2^k$$

(Here A and B are square matrices of the same size and k is a whole number.)

Problem 6. [20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

a) A 3×2 matrix A such that $Ax = (1, 2, 3)$ has more than one least-squares solution.

Any 3×2 matrix that does not have full column rank is an example.

b) A matrix A in RREF satisfying $\dim \mathrm{Row}(A) = 2$ and $\dim \mathrm{Nul}(A) = 3$.

Any RREF matrix with five columns and two nonzero rows is an example.

c) A matrix Q with orthonormal columns, such that $\det(QQ^T) = 0$.

Any non-square matrix with orthonormal columns is an example.

d) A matrix A whose column space $V = \mathrm{Col}(A)$ is a plane in $\mathbf{R}^3$, such that $\mathrm{rank}(P_V) = 1$.

Impossible: $\mathrm{rank}(P_V) = \dim(V) = 2$.