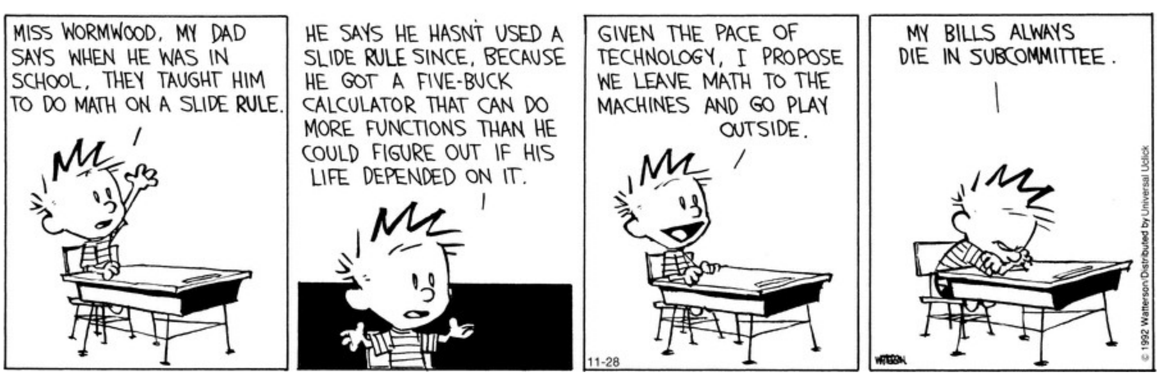
# MATH 218D-1
# MIDTERM EXAMINATION 1

| Name | | Duke NetID | |
|---|---|---|---|

Please **read all instructions** carefully before beginning.

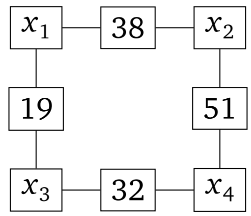
- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a $3 \times 5$**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



## Problem 1.

[20 points]

Consider the following square:




Our *problem* is to find values for the unknowns $x_1, x_2, x_3, x_4$ such that the sum of the numbers along each edge equals 100.

**a)** Express this problem as a system of linear equations.

**b)** Express this problem as a matrix equation $Ax = b$.

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

**c)** Compute the reduced row echelon form of $\left(A \mid b\right)$.

[Scratch work for Problem 1]

**(Problem 1, continued)**

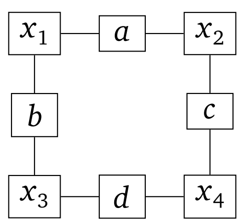
**d)** Find the general solution in parametric vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} +$$

**e)** Write one answer to the *problem*: that is, find numbers $x_1, x_2, x_3, x_4$ such that the sum of the numbers along each edge equals 100.

$x_1 =$
$x_2 =$
$x_3 =$
$x_4 =$

Let us replace the middle numbers with variables $a, b, c, d$:




**f)** This *problem* has a solution when

$$100\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \text{Span}\left\{ \qquad \qquad \right\}.$$

[Scratch work for Problem 1]

## Problem 2. [20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

and the subspace $V = \mathrm{Span}\{v_1, v_2, v_3\}$.

**a)** Find a linear relation among $\{v_1, v_2, v_3\}$.

$$\boxed{\phantom{00}}\, v_1 + \boxed{\phantom{00}}\, v_2 + \boxed{\phantom{00}}\, v_3 = 0$$

**b)** Which vectors are in the span of the other two? Fill in the bubbles of all that apply.

○ $v_1$ ○ $v_2$ ○ $v_3$

**c)** Find a basis for $V$.

$$\left\{ \qquad\qquad \right\}$$

**d)** $\dim(V) = \boxed{\phantom{00}}$.

**e)** The vector $(1, 4, -1)$ is in $V$. Express $(1, 4, -1)$ as a linear combination of your basis vectors in **c)**.

$$\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} =$$

**f)** Which of the following is an implicit equation for $V$? Fill in the bubble for the correct answer

○ $x_1 + x_2 + x_3 = 4$ ○ $3x_1 - x_2 - x_3 = 0$

○ $x_1 + x_2 - 3x_3 = 0$ ○ $x_1 + x_2 + x_3 = 0$

[Scratch work for Problem 2]

## Problem 3. [10 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 5 \\ 5 & -5 & -1 \\ -2 & 2 & 1 \end{pmatrix}.$$

The reduced row echelon form of $\left(A \mid I_3\right)$ is given below:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 5 & 1 & 0 & 0 \\ 5 & -5 & -1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 & 2/3 & 5/3 \\ 0 & 0 & 0 & 1 & -10/3 & -25/3 \end{array}\right).$$

**a)** Find bases for all four fundamental subspaces:

$\mathrm{Nul}(A)$: $\left\{ \quad \right\}$ $\mathrm{Col}(A)$: $\left\{ \quad \right\}$

$\mathrm{Row}(A)$: $\left\{ \quad \right\}$ $\mathrm{Nul}(A^T)$: $\left\{ \quad \right\}$

**b)** Explain why $A$ is not invertible.

[Scratch work for Problem 3]

## Problem 4. [10 points]

Consider the matrix

$$B = \begin{pmatrix} 0 & 5 & 2 \\ 5 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix}.$$

The reduced row echelon form of $\left(B \mid I_3\right)$ is given below:

$$\left(\begin{array}{ccc|ccc} 0 & 5 & 2 & 1 & 0 & 0 \\ 5 & -1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & -8 \\ 0 & 1 & 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 & -10 & -25 \end{array}\right).$$

**a)** Find bases for all four fundamental subspaces:

$\mathrm{Nul}(A)$: $\left\{ \quad \right\}$ $\mathrm{Col}(A)$: $\left\{ \quad \right\}$

$\mathrm{Row}(A)$: $\left\{ \quad \right\}$ $\mathrm{Nul}(A^T)$: $\left\{ \quad \right\}$

**b)** The inverse of $B$ is

$$B^{-1} = \begin{pmatrix} \quad \end{pmatrix}$$

**c)** Solve the following matrix equations:

$$Bx = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad By = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Bz = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

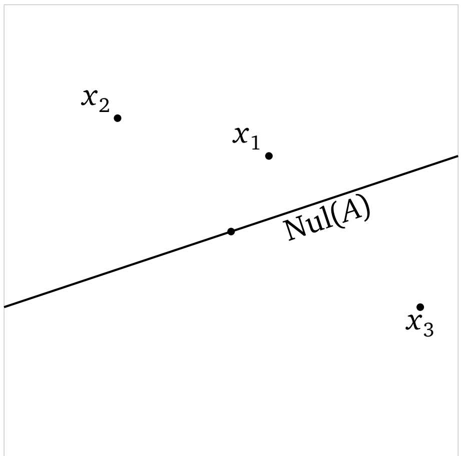
$$x = \begin{pmatrix} \quad \end{pmatrix} \qquad y = \begin{pmatrix} \quad \end{pmatrix} \qquad z = \begin{pmatrix} \quad \end{pmatrix}$$

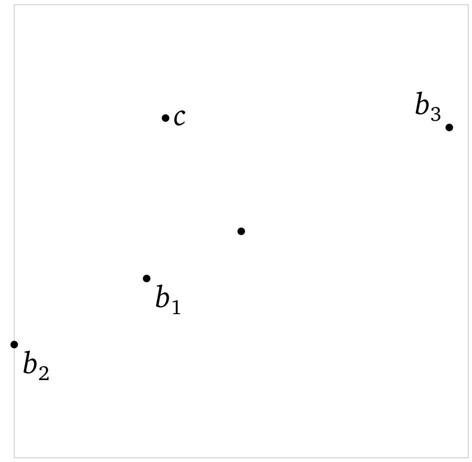
[Scratch work for Problem 4]

## Problem 5.

[8 points]

The null space of a certain $2 \times 2$ matrix $A$ is drawn in the plot on the left. For certain vectors $x_1, x_2, x_3$ drawn on the left, the vectors $b_1 = Ax_1$, $b_2 = Ax_2$, $b_3 = Ax_3$ are drawn on the right. Another vector $c$ is also drawn on the right.




**a)** $\operatorname{rank}(A) = \square$.

**b)** Draw Col($A$) on the right.

**c)** Draw *and label* the solution sets of $Ax = b_1$, $Ax = b_2$, and $Ax = b_3$ on the left.

**d)** What is the solution set of $Ax = c$?

[Scratch work for Problem 5]

## Problem 6.

[16 points]

Short-answer questions: no justification is necessary.

**a)** Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{pmatrix}.$$

Which of the following vectors are in $\mathrm{Nul}(A^T)$? Fill in the bubbles of all that apply.

○ $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

**b)** Which of the following subspaces are equal to the plane $\{(x,y,0) \,:\, x,y \in \mathbf{R}\}$ (the $xy$-plane in $\mathbf{R}^3$)? Fill in the bubbles of all that apply.

○ $\mathrm{Nul}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

○ $\mathrm{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

○ $\mathrm{Nul}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

○ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right\}$

○ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

○ $\mathrm{Col}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

**c)** If $A$ is a $20 \times 60$ matrix, then

$$\mathrm{rank}(A) + \dim \mathrm{Nul}(A^T) = \boxed{\phantom{00}}.$$

**d)** Suppose that $A$ is a $2 \times 3$ matrix whose null space is a line. What can you conclude about $A$? Fill in the bubbles of all that apply.

○ $A$ has full column rank

○ $A$ has full row rank

○ $Ax = b$ is consistent for every $b \in \mathbf{R}^2$

○ $A$ is invertible

○ $\mathrm{Col}(A) = \mathbf{R}^2$

○ $\mathrm{Row}(A)$ is a plane in $\mathbf{R}^3$

○ $A$ has linearly independent columns

○ The solution set of $Ax = \binom{1}{1}$ is a line.

[Scratch work for Problem 6]

## Problem 7. [16 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

**a)** A $3 \times 4$ matrix with linearly independent columns.

**b)** A *non-invertible* $2 \times 2$ matrix whose null space is $\{0\}$.

**c)** A $2 \times 2$ matrix whose column space is equal to its null space.

**d)** A $3 \times 3$ matrix whose column space is equal to its null space.

[Scratch work for Problem 7]