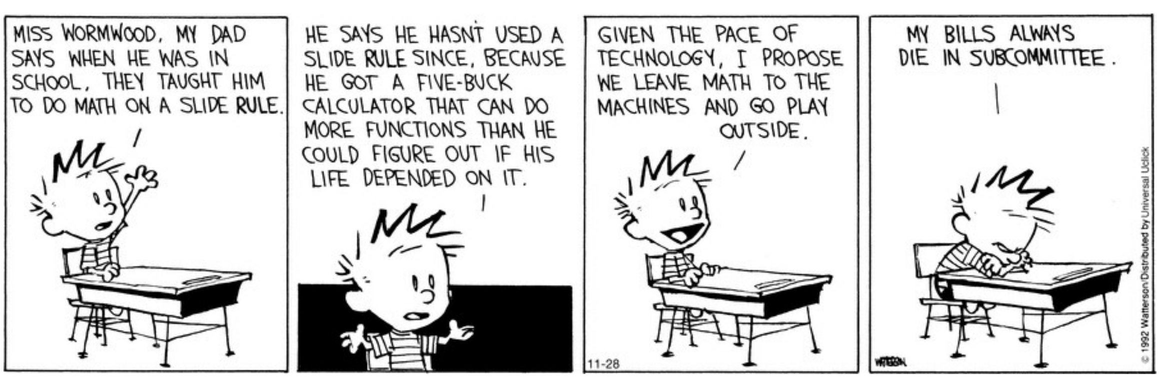
MATH 218D-1
MIDTERM EXAMINATION 1

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

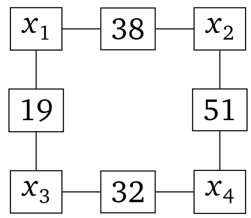
- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1.

[20 points]

Consider the following square:



Our *problem* is to find values for the unknowns x_1, x_2, x_3, x_4 such that the sum of the numbers along each edge equals 100.

a) Express this problem as a system of linear equations.

$$\begin{cases} x_1 + x_2 = 100 - 38 \\ x_1 + x_3 = 100 - 19 \\ x_2 + x_4 = 100 - 51 \\ x_3 + x_4 = 100 - 32 \end{cases}$$

b) Express this problem as a matrix equation $Ax = b$.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 62 \\ 81 \\ 49 \\ 68 \end{pmatrix}$$

c) Compute the reduced row echelon form of $\left(A \mid b\right)$.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 13 \\ 0 & 1 & 0 & 1 & 49 \\ 0 & 0 & 1 & 1 & 68 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

d) Find the general solution in parametric vector form.

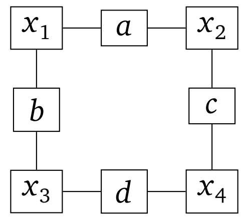
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 49 \\ 68 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

e) Write one answer to the *problem*: that is, find numbers x_1, x_2, x_3, x_4 such that the sum of the numbers along each edge equals 100.

Substitute any value for x_4 in **d)**. For instance, taking $x_4 = 1$ gives

$$\begin{aligned} x_1 &= 14 \\ x_2 &= 48 \\ x_3 &= 67 \\ x_4 &= 1 \end{aligned}$$

Let us replace the middle numbers with variables a, b, c, d:



f) This *problem* has a solution when

$$100\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\in \text{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}.$$

Problem 2.

[20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

and the subspace $V = \mathrm{Span}\{v_1, v_2, v_3\}$.

a) Find a linear relation among $\{v_1, v_2, v_3\}$.

$$\boxed{2}\, v_1 + \boxed{-3}\, v_2 + \boxed{1}\, v_3 = 0$$

b) Which vectors are in the span of the other two? Fill in the bubbles of all that apply.

● v_1 ● v_2 ● v_3

c) Find a basis for V.

There are many correct answers. You probably chose this one: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

d) $\dim(V) = \boxed{2}$.

e) The vector $(1, 4, -1)$ is in V. Express $(1, 4, -1)$ as a linear combination of your basis vectors in **c)**.

This depends on your answer to **c)**. $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

f) Which of the following is an implicit equation for V? Fill in the bubble for the correct answer

○ $x_1 + x_2 + x_3 = 4$ ● $3x_1 - x_2 - x_3 = 0$

○ $x_1 + x_2 - 3x_3 = 0$ ○ $x_1 + x_2 + x_3 = 0$

Problem 3.

[10 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 5 \\ 5 & -5 & -1 \\ -2 & 2 & 1 \end{pmatrix}.$$

The reduced row echelon form of $\left(A \mid I_3 \right)$ is given below:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 5 & 1 & 0 & 0 \\ 5 & -5 & -1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 & 2/3 & 5/3 \\ 0 & 0 & 0 & 1 & -10/3 & -25/3 \end{array}\right).$$

a) Find bases for all four fundamental subspaces:

Nul(A): $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Col(A): $\left\{ \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \right\}$

Row(A): $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Nul(A^T): $\left\{ \begin{pmatrix} -3 \\ 10 \\ 25 \end{pmatrix} \right\}$

b) Explain why A is not invertible.

It only has 2 pivots. (There are many other reasons.)

Problem 4. [10 points]

Consider the matrix

$$B = \begin{pmatrix} 0 & 5 & 2 \\ 5 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix}.$$

The reduced row echelon form of $\left(B \mid I_3 \right)$ is given below:

$$\left(\begin{array}{ccc|ccc} 0 & 5 & 2 & 1 & 0 & 0 \\ 5 & -1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & -8 \\ 0 & 1 & 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 & -10 & -25 \end{array}\right).$$

a) Find bases for all four fundamental subspaces:

Nul(A): $\{\}$ Col(A): $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Row(A): $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Nul(A^T): $\{\}$

b) The inverse of B is

$$B^{-1} = \begin{pmatrix} 1 & -3 & -8 \\ -1 & 4 & 10 \\ 3 & -10 & -25 \end{pmatrix}$$

c) Solve the following matrix equations:

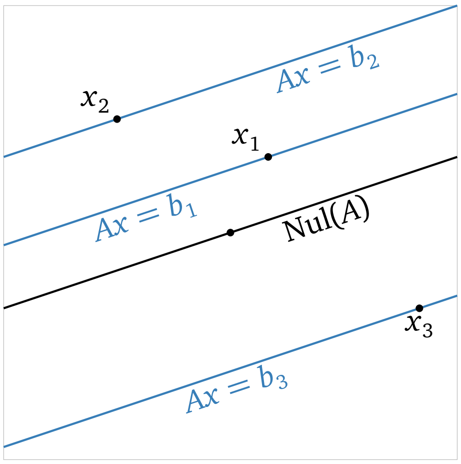
$$Bx = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad By = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Bz = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

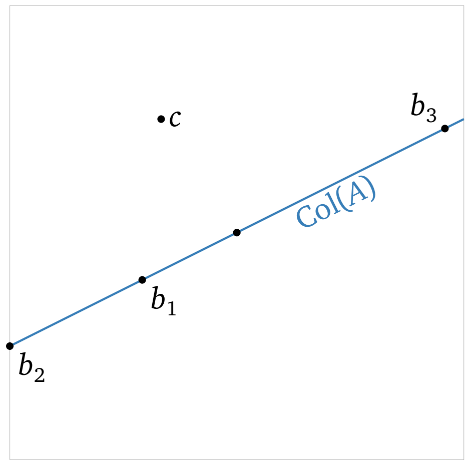
$$x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \qquad y = \begin{pmatrix} -3 \\ 4 \\ -10 \end{pmatrix} \qquad z = \begin{pmatrix} -8 \\ 10 \\ -25 \end{pmatrix}$$

Problem 5.

[8 points]

The null space of a certain 2×2 matrix A is drawn in the plot on the left. For certain vectors x_1, x_2, x_3 drawn on the left, the vectors $b_1 = Ax_1$, $b_2 = Ax_2$, $b_3 = Ax_3$ are drawn on the right. Another vector c is also drawn on the right.





a) $\text{rank}(A) = \boxed{1}$.

b) Draw Col(A) on the right.

c) Draw *and label* the solution sets of $Ax = b_1$, $Ax = b_2$, and $Ax = b_3$ on the left.

d) What is the solution set of $Ax = c$?

It is empty because $c \notin \text{Col}(A)$.

Problem 6.

[16 points]

Short-answer questions: no justification is necessary.

a) Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{pmatrix}.$$

Which of the following vectors are in $\mathrm{Nul}(A^T)$? Fill in the bubbles of all that apply.

○ $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ ● $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ● $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ● $\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

b) Which of the following subspaces are equal to the plane $\{(x, y, 0) \,:\, x, y \in \mathbf{R}\}$ (the xy-plane in $\mathbf{R}^3$)? Fill in the bubbles of all that apply.

● $\mathrm{Nul}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

○ $\mathrm{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

○ $\mathrm{Nul}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

○ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right\}$

● $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

● $\mathrm{Col}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

c) If A is a 20×60 matrix, then

$$\mathrm{rank}(A) + \dim \mathrm{Nul}(A^T) = \boxed{20}.$$

d) Suppose that A is a 2×3 matrix whose null space is a line. What can you conclude about A? Fill in the bubbles of all that apply.

○ A has full column rank

● A has full row rank

● $Ax = b$ is consistent for every $b \in \mathbf{R}^2$

○ A is invertible

● $\mathrm{Col}(A) = \mathbf{R}^2$

● $\mathrm{Row}(A)$ is a plane in $\mathbf{R}^3$

○ A has linearly independent columns

● The solution set of $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a line.

Problem 7. [16 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A 3×4 matrix with linearly independent columns.

As if. A wide matrix cannot have a pivot in every column.

b) A *non-invertible* 2×2 matrix whose null space is $\{0\}$.

Yeah, right. An invertible matrix has full column rank.

c) A 2×2 matrix whose column space is equal to its null space.

One such matrix is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The column space and null space of this matrix are both the x-axis.

d) A 3×3 matrix whose column space is equal to its null space.

Not gonna happen. The dimensions of the column space and null space must sum to 3.