Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a simple calculator for doing arithmetic, but you should not need one. You may bring a 3 × 5-inch note card covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [15 points]

a) Find the $LU$ decomposition of this matrix:

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & 11 \end{pmatrix}.$$ 

$$L = \begin{pmatrix} \hline \hline \end{pmatrix} \quad U = \begin{pmatrix} \hline \hline \end{pmatrix}$$

b) Express the matrix $L$ that you computed above as a product of three elementary matrices.

$$L = \begin{pmatrix} \hline \hline \end{pmatrix} \begin{pmatrix} \hline \hline \end{pmatrix} \begin{pmatrix} \hline \hline \end{pmatrix}$$
[Scratch work for Problem 1]
c) Compute $L^{-1}$.

\[
L^{-1} = \begin{pmatrix}
\end{pmatrix}
\]

\[
L^{-1} = \begin{pmatrix}
\end{pmatrix}
\]

d) Explain why a computer would probably compute a $PA = LU$ decomposition, beginning with the row swap $R_1 \leftrightarrow R_3$.

e) Given the decomposition

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 2 & -8 & -5 \\
-1 & -5 & 2 & 0 \\
2 & 0 & 3 & 2 \\
-1 & -3 & 0 & -1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-2 & 3 & 1 & 0 \\
0 & -1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -3 & 0 & -1 \\
0 & -2 & 2 & 1 \\
0 & 0 & -3 & -3 \\
0 & 0 & 0 & 2
\end{pmatrix},
\]

solve the equation

\[
\begin{pmatrix}
0 & 2 & -8 & -5 \\
-1 & -5 & 2 & 0 \\
2 & 0 & 3 & 2 \\
-1 & -3 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
7 \\
-7 \\
2 \\
-4
\end{pmatrix}.
\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
\end{pmatrix}
\]
[Scratch work for Problem 1]
Problem 2. [20 points]

a) Compute the reduced row echelon form of the matrix

\[
\begin{pmatrix}
1 & 3 & 4 & 1 \\
-3 & -9 & -6 & -1 \\
2 & 6 & 2 & 1
\end{pmatrix}
\]

Be sure to write down all row operations that you perform.

RREF:

Now we switch matrices to avoid carry-through error. Consider the matrix \( A \) and its reduced row echelon form:

\[
A = \begin{pmatrix}
1 & -1 & 4 & -10 & 1 \\
-3 & 3 & -1 & -3 & -1 \\
2 & -2 & 2 & -2 & 1
\end{pmatrix}
\quad \xrightarrow{\text{RREF}} \quad \begin{pmatrix}
1 & -1 & 0 & 2 & 0 \\
0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

b) Circle all of the free variables in the system \( Ax = 0 \):

\[
\begin{align*}
x_1 & \\
x_2 & \\
x_3 & \\
x_4 & \\
x_5 & \\
\end{align*}
\]

c) Compute a basis for \( \text{Nul}(A) \).

basis:
(Problem 2, continued)

d) Given the identity

\[
A \begin{pmatrix} 1 \\ -3 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -15 \\ 12 \end{pmatrix},
\]

write the solution set of \( Ax = (10, -15, 12) \) as a translate of a span.

solution set: \( \begin{pmatrix} \end{pmatrix} + \text{Span} \begin{pmatrix} \end{pmatrix} \)

e) Compute a basis for \( \text{Row}(A) \).

basis:

f) Compute a basis for \( \text{Col}(A) \).

basis:

g) Compute a basis for \( \text{Col}(A) \) consisting of vectors with all coordinates equal to 0 or 1.

basis:

h) Compute a basis for \( \text{Nul}(A^T) \).

basis:
[Scratch work for Problem 2]
Problem 3.  

The matrix
\[ A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ 2 & -1 & -7 \end{pmatrix} \] has null space \( \text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\} \).

a) Find a linear relation among the columns of \( A \).
\[ \square \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \square \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \square \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} = 0 \]

b) \( \text{rank}(A) = \square \)

c) Which of the following sets form a basis for \( \text{Col}(A) \)? Circle all that apply.
\[ \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \end{pmatrix} \right\} \]

d) \( \text{Row}(A) \) is a (circle one) point / line / plane / space in \( \mathbb{R}^\square \).

e) Which of the following sets form a basis for \( \text{Nul}(A^\top) \)? Circle all that apply.
\[ \left\{ \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} \right\}, \left\{ \right\} \]
[Scratch work for Problem 3]
Problem 4. [10 points]

Consider the matrix \( A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix} \).

a) Compute bases for all four fundamental subspaces of \( A \).

\[ \text{Col}(A): \begin{cases} \text{Row}(A): \end{cases} \]

\[ \text{Nul}(A^T): \begin{cases} \text{Nul}(A): \end{cases} \]

b) Draw and label \( \text{Row}(A) \) and \( \text{Nul}(A) \) in the grid on the left, and \( \text{Col}(A) \) and \( \text{Nul}(A^T) \) in the grid on the right. Be precise!

c) Draw the solution set of \( Ax = \begin{pmatrix} -4 \\ 8 \end{pmatrix} \) in the grid on the left.
[Scratch work for Problem 4]
Problem 5. [20 points]

Short-answer questions: no justification is necessary unless indicated otherwise.

a) If $A$ is a $5 \times 2$ matrix with full column rank, which of the following statements must be true about $A$? Fill in the bubbles of all that apply.

- $\text{rank}(A) = 5$
- $\text{Col}(A)$ is a plane in $\mathbb{R}^5$
- $\text{Nul}(A) = \{}$
- $Ax = b$ has a unique solution for every $b \in \mathbb{R}^5$
- $Ax = 0$ has a unique solution
- $\text{Nul}(A^T)$ is a plane in $\mathbb{R}^5$
- $\text{Row}(A) = \mathbb{R}^2$

b) A certain $3 \times 3$ matrix $A$ has null space equal to $\text{Span}\{(1, 1, 1)\}$. Which of the following sets is necessarily equal to the solution set of $Ax = b$ for some vector $b \in \mathbb{R}^3$? Fill in the bubbles of all that apply.

- $\text{Span}\{(1, 1, 1)\}$
- $\{\}$
- $\{(1, 1, 1)\}$
- $\{(t, t, 1) : t \in \mathbb{R}\}$
- $\{(t, t, 1 + t) : t \in \mathbb{R}\}$
- $(11, 2, -1) + \text{Span}\{(1, 1, 1)\}$

c) Is this set a subspace?

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 0\}$$

If so, express $V$ as the null space or the column space of a matrix. If not, explain why not.

d) A certain $2 \times 2$ matrix

$$A = \begin{pmatrix} | & | \\ v & w \end{pmatrix}$$

has columns $v$ and $w$, pictured below. Solve the equation $Ax = b$, where $b$ is the vector in the picture.
[Scratch work for Problem 5]
Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A $3 \times 3$ matrix whose row space and null space are both planes in $\mathbb{R}^3$.

b) A nonzero $2 \times 2$ matrix whose column space is contained in its null space.

c) A $3 \times 3$ matrix $A$ such that $\dim \text{Col}(A) = \dim \text{Nul}(A)$.

d) A $3 \times 3$ matrix of rank 2 whose null space is equal to its left null space.
[Scratch work for Problem 6]