Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a simple calculator for doing arithmetic. You may bring a 8.5 × 11-inch note sheet covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.
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Problem 1. [25 points]

Consider the matrix

\[ A = \begin{pmatrix} -2/3 & 8/3 & 2/3 \\ 2 & 2 & 3 \end{pmatrix}. \]

a) The rank of \( A \) is \( r = \square \).

b) Explain why it is better to orthogonally diagonalize \( AA^T \) instead of \( A^T A \) when computing the SVD of \( A \).

Here is the matrix \( AA^T \):

\[ AA^T = \begin{pmatrix} 8 & 6 \\ 6 & 17 \end{pmatrix}. \]

c) Compute the characteristic polynomial of \( AA^T \).

\[ p(\lambda) = \]

d) Compute the eigenvalues of \( AA^T \).

\[ \lambda_1 = \square \quad \lambda_2 = \square \]

e) Compute an orthonormal eigenbasis of \( AA^T \).

\[ u_1 = \begin{pmatrix} \ \\ \ \\ \end{pmatrix} \quad u_2 = \begin{pmatrix} \ \\ \ \\ \end{pmatrix} \]
[Scratch work for Problem 1]
(Problem 1, continued)

f) Compute the SVD of $A$ in outer product form.

$$A = \begin{pmatrix} \vdots \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}^T + \begin{pmatrix} \vdots \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}^T$$


g) Compute the SVD of $A$ in matrix form: $A = U\Sigma V^T$ for

$$U = \begin{pmatrix} \vdots \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \vdots \end{pmatrix}, \quad V = \begin{pmatrix} \vdots \end{pmatrix}$$

h) What are the eigenvalues of $A^TA$?
[Scratch work for Problem 1]
Problem 2. [35 points]

Consider the following matrix and its SVD:

\[
A = \begin{pmatrix}
-2 & 4 \\
4 & 2 \\
1 & 3
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\
1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\
1/\sqrt{3} & 0 & -2/\sqrt{6}
\end{pmatrix} \begin{pmatrix}
\sqrt{30} & 0 \\
0 & 2\sqrt{5} \\
0 & 0
\end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix}
1 & -3 \\
3 & 1
\end{pmatrix}^T
\]

a) Find an orthonormal basis for \( \text{Col}(A) \).

b) Find an orthonormal basis for \( \text{Nul}(A^T) \).

c) Write the SVD of \( A \) in outer product form.

\[
A = \left( \begin{pmatrix}
\vdots
\end{pmatrix} \right) \left( \begin{pmatrix}
\vdots
\end{pmatrix} \right)^T + \left( \begin{pmatrix}
\vdots
\end{pmatrix} \right) \left( \begin{pmatrix}
\vdots
\end{pmatrix} \right)^T
\]
[Scratch work for Problem 2]
(Problem 2, continued)

d) Explain why you know $A^T A = \begin{pmatrix} \frac{21}{3} & \frac{3}{29} \end{pmatrix}$ is positive-definite without doing any calculations.

e) Find an orthogonal diagonalization of $A^T A$: that is, $A^T A = QDQ^T$ for

$$Q = \begin{pmatrix} \end{pmatrix}$$
$$D = \begin{pmatrix} \end{pmatrix}$$

f) Draw the ellipse defined by $q(x_1, x_2) = 21x_1^2 + 29x_2^2 + 6x_1x_2 = 1$. Grid lines are 0.05 units apart.

g) Find the maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$. At which vectors $x$ is the maximum achieved?

$$\text{max} = \begin{pmatrix} \end{pmatrix} \quad x = \begin{pmatrix} \end{pmatrix}$$

h) The minimum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is \boxed{\ldots}.
[Scratch work for Problem 2]
Problem 3.

Consider the following matrix whose columns consist of 3 data points with 3 measurements each:

\[
A_0 = \begin{pmatrix}
4 & 8 & 0 \\
8 & 1 & 6 \\
6 & 1 & 2 \\
\end{pmatrix}
\]

a) Compute the recentered matrix \( A \) (subtract the means of the measurements).

\[A = \begin{pmatrix}
\end{pmatrix}\]

The matrix \( A \) has SVD

\[
A = 3\sqrt{6} \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} + 3\sqrt{2} \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}
\]

and covariance matrix

\[
S = \frac{1}{2}AA^T = \begin{pmatrix} 16 & -10 & -2 \\ -10 & 13 & 8 \\ -2 & 8 & 7 \end{pmatrix}.
\]

b) The rank of \( A \) is \[\square\].

c) The total variance is \( s^2 = \square \).

d) The variance of the first measurement is \( s_1^2 = \square \).

e) The variance in the direction \( u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) is \( s(u)^2 = \square \).

f) The variance along the plane \( V = \text{Span} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}^\perp \) is \( s(V)^2 = \square \).
[Scratch work for Problem 3]
(Problem 3, continued)

**g)** The line of best fit of the recentered data points is spanned by \( u_1 = \) .

The variance along this line is \( s(u_1)^2 = \) , and the error\(^2\) (the variance along the perpendicular plane) is \( s^2 \).

**h)** The direction of second-largest variance is \( u_2 = \) , and the variance in this direction is \( s(u_2)^2 = \) .

**i)** Find the plane containing the original 3 data points (the columns of \( A_0 \)) in the form \( p + \text{Span}\{\cdots\} \).

\[
\begin{pmatrix}
\end{pmatrix}
+ \text{Span}\left\{
\begin{pmatrix}
\end{pmatrix}
\right\}
\]
[Scratch work for Problem 3]
Problem 4. [10 points]

a) Compute the characteristic polynomial of the matrix

\[ A_1 = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \]

\[ p(\lambda) = \]

b) The matrix

\[ A_2 = \begin{pmatrix} 2 & 6 & 6 \\ 3 & -13 & -30 \\ -1 & 6 & 13 \end{pmatrix} \]

has characteristic polynomial

\[ p(\lambda) = - (\lambda - 1)(\lambda - 2)(\lambda + 1). \]

Find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

\[ C = \]

\[ D = \]
[Scratch work for Problem 4]
Problem 5. [15 points]

The matrix

\[ A = \begin{pmatrix}
-1/2 & 10/3 & 25/3 \\
7/2 & -31/3 & -85/3 \\
-3/2 & 14/3 & 38/3 \\
\end{pmatrix} \]

has a factorization \( A = C D C^{-1} \) for

\[
C = \begin{pmatrix}
0 & 5 & 2 \\
5 & -1 & -2 \\
-2 & 1 & 1 \\
\end{pmatrix}, \quad D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/3 \\
\end{pmatrix}.
\]

a) Compute \( C^{-1} \).

\[
C^{-1} = \begin{pmatrix}
\ldots \\
\ldots \\
\ldots \\
\end{pmatrix}
\]

b) If \( v_0 = (3, 6, -2) \), compute \( v_k = A^k v_0 \):

\[
v_k = \begin{pmatrix}
\ldots \\
\ldots \\
\ldots \\
\end{pmatrix}
\]

c) \( \lim_{k \to \infty} A^k v_0 = \begin{pmatrix}
\ldots \\
\ldots \\
\ldots \\
\end{pmatrix} \).
[Scratch work for Problem 5]
Problem 6. [15 points]

A certain $2 \times 5$ matrix $A$ has singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$ 

The columns of $A$ are drawn as dots on the grid below, and the right singular vectors are drawn as arrows.

a) Draw the columns of $\sigma_1 u_1 v_1^T$ as dots on the grid.

b) Draw the columns of $\sigma_2 u_2 v_2^T$ as arrows on the grid.

c) Explain which geometric quantity in the picture corresponds to $\sigma_2^2$. 

\[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
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\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\]
[Scratch work for Problem 6]
Problem 7.  

[10 points]

a) Find all least-squares solutions of $Ax = b$ in parametric vector form, where

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ -1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}.$$ 

$$x = \begin{pmatrix} \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} \end{pmatrix} \right\}$$

b) Draw a picture of your answer to a) in the grid below.
[Scratch work for Problem 7]
Problem 8.  

Short-answer questions: no justification is necessary.

a) Find the \( L D L^T \) decomposition of the positive-definite, symmetric matrix

\[
A = \begin{pmatrix} 2 & 6 \\ 6 & 21 \end{pmatrix} \quad \implies \quad L = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix} \quad D = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}
\]

b) Consider the plane

\[
V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]

List three different methods you could use to compute the projection matrix \( P_V \).

c) Suppose that \( A \) is not diagonalizable and that \( A^{(2)} \equiv \begin{pmatrix} 3 \\ \end{pmatrix} \). What is the characteristic polynomial of \( A \)?

\[
p(\lambda) = \]

d) Which of the following sets form a basis for the plane

\[V = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 0\}\]?

*Fill in the bubbles of all that apply.*

- \( \bigcirc \bigg\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \bigg\} \)
- \( \bigcirc \bigg\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \bigg\} \)
- \( \bigcirc \bigg\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \bigg\} \)
- \( \bigcirc \bigg\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \bigg\} \)
- \( \bigcirc \bigg\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \bigg\} \)

\[ e) \text{ Let } A \text{ be a } 4 \times 4 \text{ matrix such that } 2 \text{ and } 1 + i \text{ are eigenvalues of } A. \text{ Suppose that the 2-eigenspace of } A \text{ is a plane. Then } \det(A) = \]

f) Suppose that the solution set of \( Ax = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) is the line \( \begin{pmatrix} 1 \\ \end{pmatrix} + \text{Span}\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \} \). What is the solution set of \( Ax = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \)?
[Scratch work for Problem 8]
Problem 9. [20 points]

True/false problems: circle the correct answer. No justification is needed.

All matrices in this problem have real entries.

a) T F If $V$ is a subspace of $\mathbb{R}^n$, then $P_V P_{V^\perp} = I_n$.

b) T F If $Ax = b$ has infinitely many solutions for every vector $b \in \mathbb{R}^m$, then $A$ has full row rank.

c) T F The maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is the same as the maximum value of $\|A^T y\|^2$ subject to $\|y\| = 1$.

d) T F If $\sigma$ is a singular value of an invertible matrix $A$, then $\sigma^{-1}$ is a singular value of $A^{-1}$.

e) T F If $A$ has linearly independent columns, then $A^T A$ is positive-semidefinite.

f) T F If $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^n$, then $n = 4$.

g) T F Every elementary matrix is diagonalizable.

h) T F A diagonalizable $5 \times 5$ matrix has 5 distinct eigenvalues.

i) T F If $A$ is an $n \times n$ matrix with linearly dependent rows, then the columns of $A$ do not span $\mathbb{R}^n$.

j) T F The diagonal entries of a square matrix are its eigenvalues.
Scratch work for Problem 9