Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a 8.5 × 11-inch **note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

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This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.
Problem 1. [25 points]

Consider the matrix
\[ A = \begin{pmatrix} -2/3 & 8/3 & 2/3 \\ 2 & 2 & 3 \end{pmatrix}. \]

a) The rank of \( A \) is \( r = 2 \).

b) Explain why it is better to orthogonally diagonalize \( AA^T \) instead of \( A^T A \) when computing the SVD of \( A \).

Because \( AA^T \) is a 2 \times 2 matrix, but \( A^T A \) is 3 \times 3.

Here is the matrix \( AA^T \):
\[ AA^T = \begin{pmatrix} 8 & 6 \\ 6 & 17 \end{pmatrix}. \]

c) Compute the characteristic polynomial of \( AA^T \).
\[ p(\lambda) = \lambda^2 - 25\lambda + 100 \]

\[ \lambda_1 = 20 \quad \lambda_2 = 5 \]

d) Compute the eigenvalues of \( AA^T \).

\[ \lambda_1 = 20 \quad \lambda_2 = 5 \]

e) Compute an orthonormal eigenbasis of \( AA^T \).
\[ u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]

f) Compute the SVD of \( A \) in outer product form.
\[ A = 2\sqrt{5} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}^T + \sqrt{5} \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}^T \]

g) Compute the SVD of \( A \) in matrix form: \( A = U\Sigma V^T \) for
\[ U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{pmatrix} \quad V = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \]

h) What are the eigenvalues of \( A^T A \)?

20, 5, and 0
Problem 2. [35 points]

Consider the following matrix and its SVD:

\[
A = \begin{pmatrix}
-2 & 4 \\
4 & 2 \\
1 & 3
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}
\end{pmatrix} \begin{pmatrix}
\sqrt{30} & 0 \\
0 & 2\sqrt{5} \\
0 & 0
\end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix}
1 & -3 \\
3 & 1
\end{pmatrix}^T
\]

a) Find an orthonormal basis for \(\text{Col}(A)\).
\[
\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}
\]

b) Find an orthonormal basis for \(\text{Nul}(A^T)\).
\[
\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}
\]

c) Write the SVD of \(A\) in outer product form.
\[
A = \sqrt{30} \begin{pmatrix}
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{pmatrix} \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}^T + 2\sqrt{5} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{pmatrix}^T
\]

d) Explain why you know \(A^T A = \begin{pmatrix} 21 & 3 \\ 3 & 29 \end{pmatrix}\) is positive-definite without doing any calculations.
\[
\text{Because } A \text{ has full column rank.}
\]

e) Find an orthogonal diagonalization of \(A^T A\): that is, \(A^T A = QDQ^T\) for
\[
Q = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}
\]
\[
D = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix}
\]
f) Draw the ellipse defined by $q(x_1, x_2) = 21x_1^2 + 29x_2^2 + 6x_1x_2 = 1$. Grid lines are 0.05 units apart.

g) Find the maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$. At which vectors $x$ is the maximum achieved?

$$\max = 30 \quad x = \pm \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

h) The minimum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is $20$. 
Problem 3.  

Consider the following matrix whose columns consist of 3 data points with 3 measurements each:

\[ A_0 = \begin{pmatrix} 4 & 8 & 0 \\ 8 & 1 & 6 \\ 6 & 1 & 2 \end{pmatrix} \]

a) Compute the recentered matrix \( A \) (subtract the means of the measurements).

\[ A = \begin{pmatrix} 0 & 4 & -4 \\ 3 & -4 & 1 \\ 3 & -2 & -1 \end{pmatrix} \]

The matrix \( A \) has SVD

\[ A = 3\sqrt{6} \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}} (1 \quad -2 \quad 1) + 3\sqrt{2} \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \quad 0 \quad -1) \]

and covariance matrix

\[ S = \frac{1}{2} AA^T = \begin{pmatrix} 16 & -10 & -2 \\ -10 & 13 & 8 \\ -2 & 8 & 7 \end{pmatrix} \]

b) The rank of \( A \) is 2.

c) The total variance is \( s^2 = 36 \).

d) The variance of the first measurement is \( s_1^2 = 16 \).

e) The variance in the direction \( u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) is \( s(u)^2 = \frac{9}{2} \).

f) The variance along the plane \( V = \text{Span} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \) is \( s(V)^2 = \frac{63}{2} \).
g) The line of best fit of the recentered data points is spanned by \( u_1 = (-2, 2, 1) \).

The variance along this line is \( s(u_1)^2 = 27 \), and the error\(^2\) (the variance along the perpendicular plane) is \( 9 \).

h) The direction of second-largest variance is \( u_2 = (2, 1, 2) \), and the variance in this direction is \( s(u_2)^2 = 9 \).

i) Find the plane containing the original 3 data points (the columns of \( A_0 \)) in the form \( p + \text{Span}\{\cdots\} \).

\[
\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}
\]
Problem 4.  [10 points]

a) Compute the characteristic polynomial of the matrix

\[ A_1 = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \]

\[ p(\lambda) = -\lambda^3 + 4\lambda^2 + 3\lambda - 5 \]

b) The matrix

\[ A_2 = \begin{pmatrix} 2 & 6 & 6 \\ 3 & -13 & -30 \\ -1 & 6 & 13 \end{pmatrix} \]

has characteristic polynomial

\[ p(\lambda) = -(\lambda - 1)(\lambda - 2)(\lambda + 1). \]

Find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

\[ C = \begin{pmatrix} -6 & 5 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

Problem 5.  [15 points]

The matrix

\[ A = \begin{pmatrix} -1/2 & 10/3 & 25/3 \\ 7/2 & -31/3 & -85/3 \\ -3/2 & 14/3 & 38/3 \end{pmatrix} \]

has a factorization \( A = CDC^{-1} \) for

\[ C = \begin{pmatrix} 0 & 5 & 2 \\ 5 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}. \]

a) Compute \( C^{-1} \).

\[ C^{-1} = \begin{pmatrix} 1 & -3 & -8 \\ -1 & 4 & 10 \\ 3 & -10 & -25 \end{pmatrix} \]

b) If \( v_0 = (3, 6, -2) \), compute \( v_k = A^k v_0 \):

\[ v_k = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} + \frac{1}{2^k} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3^k} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \]

\[ \lim_{k \to \infty} A^k v_0 = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}. \]
Problem 6. [15 points]

A certain $2 \times 5$ matrix $A$ has singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$  

The columns of $A$ are drawn as dots on the grid below, and the right singular vectors are drawn as arrows.

a) Draw the columns of $\sigma_1 u_1 v_1^T$ as dots on the grid.

b) Draw the columns of $\sigma_2 u_2 v_2^T$ as arrows on the grid.

c) Explain which geometric quantity in the picture corresponds to $\sigma_2^2$.
   
   It is the sum of the squares of the lengths of the arrows from b).
a) Find all least-squares solutions of $Ax = b$ in parametric vector form, where

\[
A = \begin{pmatrix}
  1 & 2 \\
  1 & 2 \\
 -1 & -2
\end{pmatrix} \quad b = \begin{pmatrix}
  6 \\
  0 \\
  3
\end{pmatrix}.
\]

\[
x = \begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}
  -2 \\
  1
\end{pmatrix}\right\}
\]

b) Draw a picture of your answer to a) in the grid below.
Problem 8. [30 points]

Short-answer questions: no justification is necessary.

a) Find the $LDL^T$ decomposition of the positive-definite, symmetric matrix

$$A = \begin{pmatrix} 2 & 6 \\ 6 & 21 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

b) Consider the plane

$$V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

List three different methods you could use to compute the projection matrix $P_V$.

Here are a few: (i) The horrible formula. (ii) Compute $P_V \perp$ by projecting onto the line spanned by $(1, 2, 3) \times (1, 1, 0)$. (iii) Compute a QR decomposition and use $P_V = QQ^T$.

c) Suppose that $A$ is not diagonalizable and that $A(2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. What is the characteristic polynomial of $A$?

$$p(\lambda) = (\lambda - 2)^2$$

d) Which of the following sets form a basis for the plane

$$V = \{ (x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 0 \}?$$

Fill in the bubbles of all that apply.

- $\bigcirc \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix}$
- $\blacklozenge \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$
- $\blacklozenge \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

- $\bigcirc \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$
- $\bigcirc \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\bigcirc \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

e) Let $A$ be a $4 \times 4$ matrix such that $2$ and $1 + i$ are eigenvalues of $A$. Suppose that the $2$-eigenspace of $A$ is a plane. Then $\det(A) = \boxed{8}$.

f) Suppose that the solution set of $Ax = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ is the line $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{Span}\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \}$. What is the solution set of $Ax = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$?

$$-\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{Span}\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \}$$
Problem 9.

True/false problems: circle the correct answer. No justification is needed.

All matrices in this problem have real entries.

a) T F If $V$ is a subspace of $\mathbb{R}^n$, then $P_V P_{V^\perp} = I_n$.

b) T F If $Ax = b$ has infinitely many solutions for every vector $b \in \mathbb{R}^m$, then $A$ has full row rank.

c) T F The maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is the same as the maximum value of $\|A^T y\|^2$ subject to $\|y\| = 1$.

d) T F If $\sigma$ is a singular value of an invertible matrix $A$, then $\sigma^{-1}$ is a singular value of $A^{-1}$.

e) T F If $A$ has linearly independent columns, then $A^T A$ is positive-semidefinite.

f) T F If $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^n$, then $n = 4$.

g) T F Every elementary matrix is diagonalizable.

h) T F A diagonalizable $5 \times 5$ matrix has 5 distinct eigenvalues.

i) T F If $A$ is an $n \times n$ matrix with linearly dependent rows, then the columns of $A$ do not span $\mathbb{R}^n$.

j) T F The diagonal entries of a square matrix are its eigenvalues.