Math 218D-1: Homework #9

due Wednesday, November 1, at 11:59pm

1. a) Compute the determinants of the matrices in HW8#1 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!

b) Compute the determinants of the matrices in HW8#1(b) and (d) again using Sarrus’ scheme.

c) For the matrix of HW8#1(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus’ scheme. Did you get the determinant?

2. Compute

$$\det \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} - \lambda I_3$$

where \(\lambda\) is an unknown real number. Your answer will be a function of \(\lambda\).

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$ 

a) Compute the cofactor matrix \(C\) of \(A\).

b) Compute \(AC^T\). What is the relationship between \(C^T\) and \(A^{-1}\)?

4. Consider the \(n \times n\) matrix \(F_n\) with 1’s on the diagonal, 1’s in the entries immediately below the diagonal, and \(-1\)’s in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \cdots.$$ 

a) Show that det\((F_2) = 2\) and det\((F_3) = 3\).

b) Expand in cofactors to show that det\((F_n) = det(F_{n-1}) + det(F_{n-2})\).

c) Compute det\((F_4), det(F_5), det(F_6), det(F_7)\) using b).

This shows that det\((F_n)\) is the \(n\)th **Fibonacci number**. (The sequence usually starts with 1, 1, 2, 3, \ldots, so our det\((F_n)\) is the usual \(n + 1\)st Fibonacci number.)
5. Let $A$ be an $n \times n$ invertible matrix with integer (whole number) entries.
   a) Explain why $\text{det}(A)$ is an integer.
   b) If $\text{det}(A) = \pm 1$, show that $A^{-1}$ has integer entries.
   c) If $A^{-1}$ has integer entries, show that $\text{det}(A) = \pm 1$.

6. Let $V$ be a subspace of $\mathbb{R}^n$. The matrix for reflection over $V$ is
   \[ R_V = I_n - 2P_V, \]
   where $P_V = I_n - P_V$ is the projection matrix onto $V^\perp$.
   a) Suppose that $V$ is the line in the picture. Draw the vectors $R_Vx_1, R_Vx_2, R_Vx_3,$
      and $R_Vx_4$ as points in the plane.

   ![Diagram of vectors](image)

   b) Show that any reflection matrix $R_V$ is orthogonal.
      \[ \text{[Hint: Recall that } P^2_V = P_{V^\perp} = P_{V^\perp}^T. \]  
   c) Let $V$ be the plane $x + y + z = 0$. Compute $R_V$ and $\text{det}(R_V)$.
   d) Let $V$ be any plane in $\mathbb{R}^3$. Prove that $\text{det}(R_V) = -1$, as follows: choose an
      orthonormal basis $\{u_1, u_2\}$ for $V$, and let $u_3 = u_1 \times u_2$. Show that the matrix $A$
      with columns $u_1, u_2, u_3$ has determinant 1, and that $R_VA$ has determinant $-1$.
      Summary: a reflection over a plane in $\mathbb{R}^3$ has determinant $-1$.
   e) Now compute $\text{det}(R_L)$, where $L$ is the $x$-axis in $\mathbb{R}^3$. 
7. Consider the parallelepiped $P$ in $\mathbb{R}^3$ spanned by

\[ v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}. \]

a) Compute the volume of $P$ using a triple product $(v_1 \times v_2) \cdot v_3$.

b) Compute the area of each face of $P$ using cross products.

c) If the “base” of $P$ is the parallelogram spanned by $v_1$ and $v_2$ (blue in the picture), show that the height of $P$ is $\|v_3\| \sin \theta$, where $\theta$ is the angle that $v_3$ makes with the base. (Draw a simpler picture.)

d) The volume of $P$ is the area of the base of $P$ times its height. How do you reconcile c) with a)? (Remember that $\|u \cdot v\| = \|u\| \|v\| \cos(\text{the angle from } u \text{ to } v)$.)

8. Use a cross product to find an implicit equation for the plane

\[ V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}. \]

Compare HW6#7(a).

9. a) Let $v = (a, b)$ and $w = (c, d)$ be vectors in the plane, and let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$, explain how the right-hand rule determines the sign of $\det(A)$.

b) Using the identity

\[
\left[ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},
\]

explain how the right-hand rule determines the sign of a $3 \times 3$ determinant.

10. Decide if each statement is true or false, and explain why.

a) The determinant of the cofactor matrix of $A$ equals the determinant of $A$.

b) $u \times v = v \times u$.

c) If $u \times v = 0$ then $u \perp v$. 
11. For each matrix \(A\) and each vector \(v\), decide if \(v\) is an eigenvector of \(A\), and if so, find the eigenvalue \(\lambda\).

\[a) \begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \quad \left(\begin{array}{c} 1 \\ 5 \\ -2 \end{array}\right), \quad \left(\begin{array}{c} 1 \end{array}\right)\]

\[b) \begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \quad \left(\begin{array}{c} 1 \\ 1 \end{array}\right), \quad \left(\begin{array}{c} 1 \end{array}\right)\]

\[c) \begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \quad \left(\begin{array}{c} 3 \\ -2 \\ -1 \end{array}\right), \quad \left(\begin{array}{c} 1 \end{array}\right)\]

\[d) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \left(\begin{array}{c} -3 \\ 2 \\ -3 \end{array}\right), \quad \left(\begin{array}{c} 0 \end{array}\right)\]

\[e) \begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \quad \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)\]

12. For each matrix \(A\) and each number \(\lambda\), decide if \(\lambda\) is an eigenvalue of \(A\); if so, find a basis for the \(\lambda\)-eigenspace of \(A\).

\[a) \begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \quad \lambda = 1 \quad b) \begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \quad \lambda = -1\]

\[c) \begin{pmatrix} 2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix}, \quad \lambda = 3 \quad d) \begin{pmatrix} 2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix}, \quad \lambda = 2\]

\[e) \begin{pmatrix} 3 & 1 & -2 \\ -2 & 0 & 4 \\ -1 & -1 & 4 \end{pmatrix}, \quad \lambda = 2 \quad f) \begin{pmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{pmatrix}, \quad \lambda = 0\]

\[g) \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \quad \lambda = 7 \quad h) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda = 0\]

13. Suppose that \(A\) is an \(n \times n\) matrix such that \(Av = 2v\) for some \(v \neq 0\). Let \(C\) be any invertible matrix. Consider the matrices

\[a) A^{-1} \quad b) A + 2I_n \quad c) A^3 \quad d) CAC^{-1}\]

Show that \(v\) is an eigenvector of \(a)–c\) and that \(Cv\) is as eigenvector of \(d\), and find the eigenvalues.

14. Here is a handy trick for computing eigenvectors of a \(2 \times 2\) matrix.

Let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be a \(2 \times 2\) matrix with eigenvalue \(\lambda\). Explain why \(\begin{pmatrix} -b \\ a - \lambda \end{pmatrix}\) and \(\begin{pmatrix} d - \lambda \\ -c \end{pmatrix}\) are \(\lambda\)-eigenvectors of \(A\) if they are nonzero.

For which matrices \(A\) does this trick fail?
15.  
   a) Show that $A$ and $A^T$ have the same eigenvalues.
   b) Give an example of a $2 \times 2$ matrix $A$ such that $A$ and $A^T$ do not share any eigenvectors.
   c) A **stochastic matrix** is a matrix with nonnegative entries such that the entries in each column sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix.  
      [**Hint:** show that $(1, 1, \ldots, 1)$ is an eigenvector of $A^T$.]

16.  
   a) Find all eigenvalues of the matrix
      $$
      \begin{pmatrix}
      1 & -1 & 2 & 3 & 4 \\
      0 & 3 & -1 & -2 & -5 \\
      0 & 0 & 1 & 2 & 4 \\
      0 & 0 & 0 & 2 & 3 \\
      0 & 0 & 0 & 0 & -1
      \end{pmatrix}.
      $$
   b) Explain how to find the eigenvalues of any triangular matrix.

17.  Recall that an **orthogonal matrix** is a square matrix with orthonormal columns. Prove that any (real) eigenvalue of an orthogonal matrix $Q$ is $\pm 1$.

18.  Give an example of each of the following, or explain why no such example exists.
   a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.
   b) A $2 \times 2$ orthogonal matrix with no real eigenvalues.

19.  Suppose that $A$ is a square matrix such that $A^k$ is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of $A$.

20.  Decide if each statement is true or false, and explain why.
   a) If $v, w$ are eigenvectors of a matrix $A$, then so is $v + w$.
   b) An eigenvalue of $A + B$ is the sum of an eigenvalue of $A$ and an eigenvalue of $B$.
   c) An eigenvalue of $AB$ is the product of an eigenvalue of $A$ and an eigenvalue of $B$.
   d) If $Ax = \lambda x$ for some vector $x$, then $\lambda$ is an eigenvalue of $A$.
   e) A matrix with eigenvalue 0 is not invertible.
   f) The eigenvalues of $A$ are equal to the eigenvalues of a row echelon form of $A$. 