1. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that:
   (1) $\dim \text{Col}(A) + \dim \text{Nul}(A)$ is the number of columns of $A$.
   (2) $\dim \text{Row}(A) + \dim \text{Nul}(A^T)$ is the number of rows of $A$.
   (3) $\dim \text{Row}(A) = \dim \text{Col}(A)$.
   [Hint: Augment with the identity matrix so you only have to do Gauss–Jordan elimination once. Feel free to use the Sage cell on the website!]

   a) $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$  
b) $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$  
c) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$  
e) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2. Suppose that $A$ is an invertible $4 \times 4$ matrix. Find bases for its four fundamental subspaces.
   [Hint: No calculations are necessary.]

3. a) Let $A$ be a $9 \times 4$ matrix of rank 3. What are the dimensions of its four fundamental subspaces?
   b) If the left null space of a $5 \times 4$ matrix $A$ has dimension 3, what is the rank of $A$?

4. Find an example of a matrix with the required properties, or explain why no such matrix exists.
   a) The column space contains $(1, 2, 3)$ and $(4, 5, 6)$, and the row space contains $(1, 2)$ and $(2, 3)$.
   b) The column space has basis $\{(1, 2, 3)\}$, and the null space has basis $\{(3, 2, 1)\}$.
   c) The dimension of the null space is one greater than the dimension of the left null space.
   d) A $3 \times 5$ matrix whose row space equals its null space.
5. Draw the four fundamental subspaces of the following matrices, in grids like below.

   a) \[
   \begin{pmatrix}
   1 & 3 \\
   2 & 6 \\
   \end{pmatrix}
   \]

   b) \[
   \begin{pmatrix}
   1 & 0 \\
   2 & 0 \\
   \end{pmatrix}
   \]

6. For the following matrix \( A \), find the pivot positions of \( A \) and of \( A^T \). Do they have the same pivots? Do they have the same rank?

   \[
   A = \begin{pmatrix}
   1 & 2 & 3 \\
   -1 & -2 & -3 \\
   4 & 5 & 6 \\
   \end{pmatrix}
   \]

7. Find a matrix \( A \) such that

   \[\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \]

   and \[\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} .\]

What is the rank of \( A \)?

8. a) If \( \text{Col}(B) \) is contained in \( \text{Nul}(A) \), then \( AB = \) ________.

   b) Find a \( 2 \times 2 \) matrix \( A \) such that \( \text{Col}(A) = \text{Nul}(A) \). What is the rank of such a matrix? [Hint: use HW4#5.]

9. a) Show that \( \text{rank}(AB) \leq \text{rank}(A) \). [Hint: Compare HW4#6.]

   b) Show that \( \text{rank}(AB) \leq \text{rank}(B) \). [Hint: Take transposes in (a).]

10. Let \( A \) be a \( 3 \times 3 \) matrix of rank 2. Explain why \( A^2 \) is not the zero matrix. [Hint: Compare Problem 8.]

11. This problem explains why we only consider square matrices when we discuss invertibility.

   a) Show that a tall matrix \( A \) (more rows than columns) does not have a right inverse, i.e., there is no matrix \( B \) such that \( AB = I_m \).

   b) Show that a wide matrix \( A \) (more columns than rows) does not have a left inverse, i.e., there is no matrix \( B \) such that \( BA = I_n \).

   [Hint: Use Problem 9.]
12. Let $A$ be an $m \times n$ matrix. Which of the following are equivalent to the statement “$A$ has full column rank”?
   a) $\text{Nul}(A) = \{0\}$
   b) $A$ has rank $m$
   c) The columns of $A$ are linearly independent
   d) $\text{dim Row}(A) = n$
   e) The columns of $A$ span $\mathbb{R}^m$
   f) $A^T$ has full column rank

13. Let $A$ be an $m \times n$ matrix. Which of the following are equivalent to the statement “$A$ has full row rank”?
   a) $\text{Col}(A) = \mathbb{R}^m$
   b) $A$ has rank $m$
   c) The columns of $A$ are linearly independent
   d) $\text{dim Nul}(A) = n - m$
   e) The rows of $A$ span $\mathbb{R}^n$
   f) $A^T$ has full column rank

14. Consider the following vectors:
   \[ u = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}, \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]
   a) Compute the lengths $\|u\|, \|v\|,$ and $\|w\|$.
   b) Compute the lengths $\|2u\|, \|-v\|,$ and $\|3w\|$.
   c) Find the unit vectors in the directions of $u$, $v$, and $w$.
   d) Check the Schwartz inequalities $|u \cdot v| \leq \|u\| \|v\|$ and $|v \cdot w| \leq \|v\| \|w\|$.
   e) Find the angles between $u$ and $v$ and between $v$ and $w$.
   f) Find the distance from $v$ to $w$.
   g) Find unit vectors $u', v', w'$ orthogonal to $u$, $v$, $w$, respectively.

15. If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v-w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

16. a) If $v \cdot w < 0$, what does that say about the angle between $v$ and $w$?
   b) Find three vectors $u, v, w$ in the $xy$-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$. 


17. Compute a basis for the orthogonal complement of each of the following spans.

a) \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\} \)

b) \( \text{Span} \left\{ \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \end{pmatrix} \right\} \)

c) \( \text{Span} \left\{ \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \\ 7 \end{pmatrix} \right\} \)

d) \( \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \)

e) \( \text{Span} \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \)

f) \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \)

18. Compute a basis for the orthogonal complement of each the following subspaces.

a) \( \text{Col} \left( \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \right) \)

b) \( \text{Nul} \left( \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \right) \)

c) \( \text{Row} \left( \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \right) \)

d) \( \text{Nul} \left( \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \right) \)

e) \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \)

f) \( \text{Col} \left( \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right) \)

[Hint: solving a)–d) requires only one Gauss-Jordan elimination, and f) doesn’t require any work.]

19. Compute a basis for the orthogonal complement of each the following subspaces.

a) \( \{ (x, y, x) : x, y \in \mathbb{R} \} \).

b) \( \{ (x, y, z) \in \mathbb{R}^3 : x = 2y + z \} \).

c) The solution set of the system of equations \( \begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases} \)

d) \( \{ x \in \mathbb{R}^2 : Ax = 2x \} \), where \( A = \begin{pmatrix} 0 & 6 \\ 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \).

e) The subspace of all vectors in \( \mathbb{R}^3 \) whose coordinates sum to zero.

f) The intersection of the plane \( x - 2y - z = 0 \) with the \( xy \)-plane.

g) The line \( \{ (t, -t, t) : t \in \mathbb{R} \} \).

[Hint: Compare HW4#17.]
20. Construct a matrix $A$ with each of the following properties, or explain why no such matrix exists.

a) The column space contains $(0, 2, 1)$, and the null space contains $(1, -1, 2)$ and $(-1, 3, 2)$.

b) The row space contains $(0, 2, 1)$, and the null space contains $(1, -1, 2)$ and $(-1, 3, 2)$.

c) $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 0$.

d) A $2 \times 2$ matrix $A$ with no zero entries such that every row of $A$ is orthogonal to every column.

e) The sum of the columns of $A$ is $(0, 0, 0)$, and the sum of the rows of $A$ is $(1, 1, 1)$. 