

Math 218D-1: Homework #3

due Wednesday, September 20, at 11:59pm

1. Express each system of linear equations as a vector equation. For example,

$$\begin{array}{r} x_1 + 2x_2 = 3 \\ -x_1 - x_2 = 4 \end{array} \rightsquigarrow x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\text{a) } \begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 + 4x_3 = 2 \end{cases} \quad \text{b) } \begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{c) } \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array} \right)$$

2. For each matrix A and vector b , decide if the system $Ax = b$ is consistent. If so, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

$$\text{a) } A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

$$\text{e) } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

3. For each matrix A and vector b in Problem 2, find the parametric *vector* form of the general solution of $Ax = b$ (if the system is consistent). For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the dimension of the solution set?

4. a) Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?
If so, what are the weights?

- b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 2. This does not require any additional computation.]

5. The equation $x + 2y = z$ determines a plane in \mathbf{R}^3 . (This is an *implicit equation* for the plane).
- What is the coefficient matrix A for this system?
 - Which are the free variables?
 - Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.
 - Do the same for the plane defined by $2y = z$. What is different?

6. The equations

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line \mathbf{R}^3 . (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t . (Use the free variable as the parameter t .)

7. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \quad t \in \mathbf{R}.$$

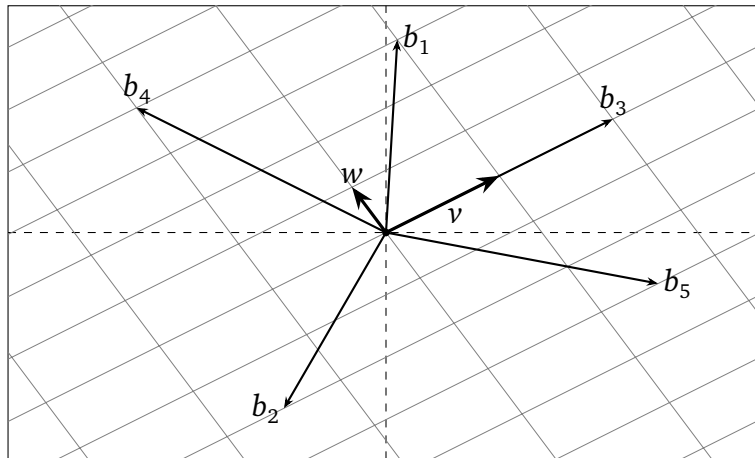
8. Suppose that A is a 3×3 matrix and b is a vector such that the solution set of $Ax = b$ is a line in \mathbf{R}^3 . How many pivots does A have?
9. Decide if each statement is true or false, and explain why.
- A square matrix has no free variables.
 - An invertible matrix has no free variables.
 - An $m \times n$ matrix has at most m pivots.
 - A wide matrix (more columns than rows) must have a free variable.
 - If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

10. Consider the vectors

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ ($c, d = -1, 0, 1, 2$) as *points* in the xy -plane.

11. Certain vectors v, w in \mathbf{R}^2 are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w . *Do not try to guess the coordinates of v and w !* This is a question about the geometry of linear combinations.



12. Consider the vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \leq a \leq 1$ and $0 \leq b \leq 1$. (This will be a shaded region in the xy -plane.)

13. Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} x = b$$

is consistent. [Hint: the answer is a span!]

14. For each matrix A and vector b , and express the solution set in the form

$$p + \text{Span}\{???\}$$

for some vector p . For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

[Hint: You found the parametric vector form in Problem 3.]

a) $A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$

15. For each matrix A in Problem 14, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution?

[Hint: this problem requires no additional computation.]

16. When is the following system consistent?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= b_1 \\ -4x_1 - 5x_2 + 5x_3 &= b_2 \\ 6x_1 + x_2 + 12x_3 &= b_3 \end{aligned}$$

Your answer should be a single linear equation in b_1, b_2, b_3 . [Hint: perform Gaussian elimination.]

Explain the relationship between this equation and

$$\text{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}.$$

17. Let A be a 3×4 matrix whose columns span the plane $x + y + z = 0$.

a) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ consistent.

b) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ inconsistent.

[Hint: this problem requires no computations at all.]

18. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

19. Give geometric descriptions of the following spans (line, plane, ...).

a) $\text{Span} \left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ b) $\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ c) $\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -6 \end{pmatrix} \right\}$

d) $\text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$ e) $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

[Hint: for d), compare Problem 16.]

20. a) List five nonzero vectors contained in $\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$.

b) Is $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ contained in $\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$?

If so, express $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

c) Show that $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ is contained in $\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$.

d) Describe $\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ geometrically.

e) Find a vector not contained in $\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$.

- 21.** Decide if each statement is true or false, and explain why.
- a)** A vector b is a linear combination of the columns of A if and only if $Ax = b$ has a solution.
 - b)** There is a matrix A such that $Ax = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ has infinitely many solutions and $Ax = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ has exactly one solution.
 - c)** The zero vector is contained in every span.
 - d)** The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what A is.
 - e)** If the zero vector is a solution of a system of equations, then the system is homogeneous.
 - f)** If $Ax = b$ has a unique solution, then A has a pivot in every column.
 - g)** If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.
 - h)** It is possible for $Ax = b$ to have exactly 13 solutions.