1. Consider the vectors
\[ u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}. \]

a) Compute \( u + v + w \) and \( u + 2v - w \).

b) Find numbers \( x \) and \( y \) such that \( w = xu + yv \).

c) Explain why every linear combination of \( u, v, w \) is also a linear combination of \( u \) and \( v \) only.

d) The sum of the coordinates of any linear combination of \( u, v, w \) is equal to \( ? \).

e) Find a vector in \( \mathbb{R}^3 \) that is not a linear combination of \( u, v, w \).

2. Find two different triples \((x, y, z)\) such that \[
x \begin{pmatrix} 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.
\]
How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector \( \frac{1}{2} v \) is a linear combination of \( v \) and \( w \).

b) \[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

c) If \( v, w \) are two vectors in \( \mathbb{R}^2 \), then any other vector \( b \) in \( \mathbb{R}^2 \) is a linear combination of \( v \) and \( w \).

4. Suppose that \( v \) and \( w \) are unit vectors: that is, \( v \cdot v = 1 \) and \( w \cdot w = 1 \). Compute the following dot products using the algebra of dot products (your answers will be actual numbers):

a) \( v \cdot (-v) \)  

b) \( (v + w) \cdot (v - w) \)  

c) \( (v + 2w) \cdot (v - 2w) \).

5. Two vectors \( v \) and \( w \) are orthogonal if \( v \cdot w = 0 \). Find nonzero vectors \( v \) and \( w \) in \( \mathbb{R}^3 \) that are orthogonal to \( (1, 1, 1) \) and to each other.
6. Compute the following matrix-vector products using both the by-row and by-column methods. If the product is not defined, explain why.

\[
\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}
\]
\[
\begin{pmatrix} 7 \\ 4 \\ -2 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 6 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 2 \\ 6 \\ -1 \end{pmatrix}
\]

7. Suppose that \( u = (x, y, z) \) and \( v = (a, b, c) \) are vectors satisfying \( 2u + 3v = 0 \). Find a nonzero vector \( w \) in \( \mathbb{R}^2 \) such that

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

8. Consider the matrices

\[
A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}
\]
\[
D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \quad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.
\]

Compute the following expressions. If the result is not defined, explain why.

a) \(-3A\)  b) \(B - 3A\)  c) \(AC\)  d) \(B^2\)

e) \(A + 2B\)  f) \(C - E\)  g) \(EB\)  h) \(D^2\)

9. Compute the product

\[
\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}
\]

in two ways:

a) using the column form, and

b) using the outer product form.

10. Consider the matrices

\[
A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.
\]

What value(s) of \( h \), if any, will make \( AB = BA \)?
11. Show that \((A + B)^2 \neq A^2 + 2AB + B^2\) when
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.
\]
What is the correct formula?
\[
(A + B)^2 = A^2 + B^2 + ______
\]
[Hint: distribute the product \((A + B)(A + B)\).]

12. In the following, find the 2 \times 2 matrix \(A\) that acts in the specified manner.
   a) \(A^{(x)} = (x)\): the identity matrix does not change the vector.
   b) \(A^{(x)} = (y)\): this is a flip over the line \(y = x\).
   c) \(A^{(x)} = (y)\): this rotates vectors clockwise by 90°.
   d) \(A^{(x)} = -(x)\): this rotates vectors by 180°.
   e) \(A^{(x)} = (0)\): this projects onto the \(y\)-axis.
   f) \(A^{(x)} = (0)\): this projects onto the \(x\)-axis.
   g) \(A^{(x)} = (y, y-2x)\): this performs the row operation \(R_2 \rightarrow 2R_1\).
   [Hint: compute \(A^{(x)}\) and \(A^{(y)}\).]

13. Consider the matrices
\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.
\]
Compute \(AD\) and \(DA\). Explain how the columns or rows of \(A\) change when \(A\) is multiplied by the diagonal matrix \(D\) on the right or the left.

14. Let \(A\) be a 4 \times 3 matrix satisfying
\[
Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \quad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \quad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.
\]
What is \(A\)?
15. Suppose that $A$ is a $4 \times 3$ matrix such that

\[
A \begin{pmatrix} 1 \\ 0 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}.
\]

Compute $Ax$, where $x$ is the vector $(3, -2, -2)$. 
[HINT: express $(3, -2, -2)$ as a linear combination of $(1, 0, 2)$ and $(0, 1, 4)$.

16. For the following matrices $A$ and $B$, compute $AB, A^T, B^T, B^T A^T$, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute $A^T B^T$?

\[
A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.
\]

17. Recall that a matrix $A$ is symmetric if $A^T = A$. Decide if each statement is true or false, and explain why.

a) If $A$ and $B$ are symmetric of the same size, then $AB$ is symmetric.

b) If $A$ is symmetric, then $A^3$ is symmetric.

c) If $A$ is any matrix, then $A^T A$ is symmetric.

18. Consider the following system of equations:

\[
x_1 - 2x_2 + x_3 = 1 \\
-2x_1 + 5x_2 + 5x_3 = 2 \\
3x_1 - 7x_2 - 7x_3 = 2.
\]

a) Use row operations to eliminate $x_1$ from all but the first equation.

b) Use row operations to modify the system so that $x_2$ only appears in the first and second equations (and $x_1$ still only appears in the first).

c) Solve for $x_3$, then for $x_2$, then for $x_1$. What is the solution?
19. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries of the table.

<table>
<thead>
<tr>
<th>System of Equations</th>
<th>Matrix Equation</th>
<th>Augmented Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x_1 + 2x_2 + 4x_3 = 9$</td>
<td>$\begin{pmatrix} 3 &amp; -5 &amp;</td>
<td>x_1 \end{pmatrix} = \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$-x_1 + 4x_3 = 2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

20. Which of the following matrices are not in row echelon form? Why not?

\[
\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}
\]

21. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

\[
\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}
\]

22. Find values of $a$ and $b$ such that the following system has a) zero, b) exactly one, and c) infinitely many solutions.

\[
\begin{align*}
2x + ay &= 4 \\
x - y &= b
\end{align*}
\]

23. Give examples of matrices $A$ in row echelon form for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on $b$

b) $\infty$ for every $b$

c) 0 or $\infty$, depending on $b$

d) 1 for every $b$.

Is there a square matrix satisfying b)? Why or why not?