Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a simple calculator for doing arithmetic, but you should not need one. You may bring a $3 \times 5$-inch note card covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!
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Problem 1. [18 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

a) The row space of $A$ is a (circle one) \(\text{line} \) \(\text{plane} \) \(\text{space} \) in (fill in the blank) \(\mathbb{R}^n\).

b) Compute the orthogonal projection of \(b = (3, 0, 0, 0, -1)\) onto \(\text{Row}(A)\).

\[
b_{\text{Row}(A)} = \begin{pmatrix} \hfill \\ \hfill \\ \end{pmatrix}.
\]

c) Compute the orthogonal projection of \(b = (3, 0, 0, 0, -1)\) onto \(\text{Nul}(A)\).

\[
b_{\text{Nul}(A)} = \begin{pmatrix} \hfill \\ \hfill \\ \end{pmatrix}.
\]
[Scratch work for Problem 1]
Now consider the matrix
\[ B = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 2 & 0 & -4 & -2 \end{pmatrix}. \]

d) The row space of \( B \) is a (circle one) \( \text{line} \) \( \text{plane} \) \( \text{space} \) in (fill in the blank) \( \mathbb{R}^\text{ } \).

e) Compute the orthogonal projection of \( b = (2, 0, 0, 3, -1) \) onto \( \text{Row}(B) \).

\[ b_{\text{Row}(B)} = \begin{pmatrix} \_ \\ \_ \end{pmatrix}. \]

f) Compute the projection matrix \( P_V \) for \( V = \text{Nul}(B) \).

\[ P_V = \begin{pmatrix} \_ \\ \_ \end{pmatrix}. \]

g) Find a basis for \( \text{Nul}(P_V) \).

\[ \{ \_ \_ \} \]
[Scratch work for Problem 1]
Problem 2. [17 points]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 4 & 1 \\
1 & 4 & -1 \\
1 & 2 & 5 \\
1 & 2 & 3
\end{pmatrix}
\]

Applying the Gram–Schmidt procedure to its columns gives:

\[
\begin{pmatrix}
1 \\
1 \\
-1 \\
-1
\end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ -1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
-1 \\
1 \\
-1
\end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}
\]

a) Compute the QR decomposition of \( A \).

b) Find the least-squares solution of \( Ax = (2, 0, -4, 2) \).
(Problem 2, continued)

c) Compute the orthogonal projection of \( b = (2, 0, -4, 2) \) onto \( V = \text{Col}(A) \).

\[
b_v = \begin{pmatrix} \hfill \\ \hfill \\ \hfill \end{pmatrix}
\]


d) Find vector \( v \) in \( \text{Nul}(A^T) \).

\[
v = \begin{pmatrix} \hfill \\ \hfill \\ \hfill \end{pmatrix}
\]

e) Compute the projection matrix \( P_v \) onto \( V = \text{Col}(A) \).

\[
P_v = \begin{pmatrix} \hfill \\ \hfill \\ \hfill \end{pmatrix}
\]

f) Find an eigenbasis for \( P_v \).

\[
\left\{ \begin{pmatrix} \hfill \\ \hfill \\ \hfill \end{pmatrix}, \begin{pmatrix} \hfill \\ \hfill \\ \hfill \end{pmatrix} \right\}
\]
[Scratch work for Problem 2]
Problem 3. [15 points]

The matrix

\[
A = \begin{pmatrix}
61/2 & 12 & -7/2 \\
-51 & -20 & 6 \\
75 & 30 & -8
\end{pmatrix}
\]

has eigenvectors

\[
w_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \quad w_3 = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}.
\]

a) Find the eigenvalue associated to each of these eigenvectors.

\[
\lambda_1 = \quad \lambda_2 = \quad \lambda_3 =
\]

b) Compute the characteristic polynomial of \(A\). (You need not expand a product of polynomials.)

\[
p(\lambda) =
\]

c) Find an invertible matrix \(C\) and a diagonal matrix \(D\) such that \(A = CDC^{-1}\).

\[
C = \quad D =
\]

d) If \(v = (-1, 3, 2)\), compute \(A^{100}v\). (You can write your answer in terms of \(w_1, w_2, w_3\).)

\[
A^{100}v =
\]

e) For which vectors \(u\) does \(\|A^k u\|\ not approach \(\infty\) as \(k \to \infty\)?
[Scratch work for Problem 3]
Problem 4. [10 points]

A certain $2 \times 2$ matrix $A$ has eigenvalues $0$ and $-1$, with corresponding eigenspaces drawn below.

a) Draw and label $Ax$ and $Ay$.

b) Draw and label $\text{Nul}(A)$ and $\text{Row}(A)$. (The eigenspaces are reproduced in gray.)
[Scratch work for Problem 4]
Problem 5.

Short-answer questions: no explanation is needed unless indicated otherwise.

a) Compute the area of the parallelogram. (Grid marks are one unit apart.)

\[
\text{area} = \square
\]

b) For which value(s) of \( k \), if any, is the following matrix not invertible?

\[
A = \begin{pmatrix}
1 & 0 & 3 & 2 \\
0 & 1 & k & 4 \\
2 & 1 & -1 & 2 \\
0 & 3 & 2 & 0
\end{pmatrix}
\]

\[
k = \square
\]

c) Suppose that \( A \) is an \( n \times n \) matrix with characteristic polynomial

\[p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.\]

Which of the following can you determine from this information?

- The number \( n \).
- The trace of \( A \).
- The determinant of \( A \).
- The eigenvalues of \( A \).
- Whether \( A \) is invertible.
- Whether \( A \) is diagonalizable.

d) Suppose that \( \nu \) is a 3-eigenvector of \( A \). Briefly explain why \( \nu \in \text{Col}(A) \).
Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A $2 \times 2$ non-diagonalizable matrix with eigenvalues 1 and $-1$.

b) A $2 \times 2$ matrix whose 1-eigenspace is the line $x + 2y = 0$ and whose 2-eigenspace is the line $x + 3y = 0$.

c) A $3 \times 2$ matrix $A$ and a vector $b$ such that $Ax = b$ does not have a least-squares solution.

d) A $2 \times 2$ matrix that is orthogonal but has no zero entries.
[Scratch work for Problem 6]