Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a four-function calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the matrix

\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 2 \end{pmatrix}. \]

a) Find a \( PA = LU \) decomposition of \( A \).

\[ P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \]

(There are other correct answers.)

b) Solve \( Ax = b \) for \( b = (1, 8, 4) \) using your answer to a).

\[ x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]

c) Compute \( A^{-1} \). Please write the row operations you performed.

\[ A^{-1} = \begin{pmatrix} -3 & -1/2 & 2 \\ 1 & 1/2 & -1 \\ 0 & -1/2 & 1 \end{pmatrix} \]

d) Solve \( Ax = b \) for an unknown vector \( b = (b_1, b_2, b_3) \). Your answer will be a formula in terms of \( b_1, b_2, b_3 \).

\[ x = \begin{pmatrix} -3b_1 - \frac{1}{2}b_2 + 2b_3 \\ b_1 + \frac{1}{2}b_2 - b_3 \\ -\frac{1}{2}b_2 + b_3 \end{pmatrix} \]

e) Express \( A^{-1} \) as a product of elementary matrices. (Write matrices, not row operations.)

\[ A^{-1} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

f) Express \( A \) as a product of elementary matrices.

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Problem 2. [15 points]

Consider the matrix equation $Ax = b$ for

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 3 & 9 & -2 & 8 \\ 2 & 6 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$ 

a) Find the parametric vector form of the solution set of $Ax = b$.

$$x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

b) Compute the following quantities:

$$\text{rank}(A) = 2, \quad \text{dim Nul}(A) = 2, \quad \text{dim Col}(A) = 2$$

$$\text{dim Row}(A) = 2, \quad \text{dim Nul}(A^T) = 1.$$ 

c) Find a basis for Nul($A$).

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) Given that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix},$$

express the solution set of $Ax = (2, 2, 8)$ as a translate of a span.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

e) Find a basis for Col($A$).

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \\ -2 \\ 2 \end{pmatrix} \right\}$$
Problem 3. [15 points]

Consider the vectors

\[ v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -7 \\ -7 \end{pmatrix}. \]

a) Find a linear relation among \( v_1, v_2, v_3 \).

\[ 5 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} = 0 \]

b) \( \text{Span}\{v_1, v_2, v_3\} \) is a (circle one) \( \text{line} \) \( \text{plane} \) \( \text{space} \) in (fill in the blank) \( \mathbb{R}^3 \).

c) Is \( v_4 \in \text{Span}\{v_1, v_2, v_3\} \)? If so, express \( v_4 \) as a linear combination of \( v_1, v_2, v_3 \).

\[ \begin{cases} \text{Yes} & v_4 = \boxed{} v_1 + \boxed{} v_2 + \boxed{} v_3 \\ \text{No} & \end{cases} \]

d) Is \( \{v_1, v_2, v_3, v_4\} \) linearly dependent? If so, find a linear relation among \( v_1, v_2, v_3, v_4 \).

\[ \begin{cases} \text{Yes} & 5 v_1 + 1 v_2 + 1 v_3 + 0 v_4 = 0 \\ \text{No} & \end{cases} \]

e) \( \text{dim Span}\{v_1, v_2, v_3, v_4\} = 3 \).

f) Which of the following sets form a basis for \( \mathbb{R}^3 \)? Circle all that apply.

\[ \{v_1, v_2\} \quad \{v_1, v_2, v_3\} \quad \{v_1, v_3, v_4\} \quad \{v_2, v_3, v_4\} \quad \{v_3, v_4\} \quad \{v_1, v_2, v_3, v_4\} \]
Problem 4. [10 points]

For a certain $2 \times 2$ matrix $A$ and vectors $x_1, x_2, x_3 \in \mathbb{R}^2$ drawn on the left, the vectors $b_1 = Ax_1$ and $b_2 = Ax_2 = Ax_3$ are drawn on the right. (All vectors are drawn as points.)

In what follows, it is important that $Ax_2$ is equal to $Ax_3$.

a) Draw the solution set of $Ax = b_2$ on the picture on the left.

b) Draw the solution set of $Ax = b_1$ on the picture on the left.

c) Draw $\text{Nul}(A)$ on the picture on the left.

d) $\text{rank}(A) = 1$.

e) Draw $\text{Col}(A)$ on the picture on the right.
Problem 5. [20 points]

Short-answer questions: no justification is necessary.

a) Consider the matrix

\[ A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Which of the following are true about \( A \)? Fill in the bubbles of all that apply.

- \( A \) has full row rank.
- \( A \) has full column rank.
- \( A \) is invertible.
- \( \text{Nul}(A) = \{0\} \).
- \( \text{Col}(A) = \mathbb{R}^3 \).
- There exists \( b \in \mathbb{R}^3 \) such that \( Ax = b \) is inconsistent.
- There exists \( b \in \mathbb{R}^3 \) such that the solution set of \( Ax = b \) is a point.
- The columns of \( A \) are linearly independent.
- The rows of \( A \) are linearly independent.

b) Which of the following are subspaces of \( \mathbb{R}^3 \)? Fill in the bubbles of all that apply.

- \( \text{Nul} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix} \)
- \( \text{Col} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix} \)
- \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \)
- The solution set of \( x_1 + 2x_2 - x_3 = 1 \\
\qquad 3x_1 + x_2 + x_3 = 2 \)
- \( \{ (x, y, z) \in \mathbb{R}^3 : xyz = 0 \} \)

c) Find a basis for the left null space of the matrix

\[ A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

\{ \}

d) If \( A \) is a 4 \( \times \) 5 matrix, then

\[ \dim \text{Nul}(A) + \dim \text{Col}(A) = 5 \quad \text{dim Nul}(A) + \dim \text{Row}(A) = 5. \]
Problem 6. [20 points]

Find examples of the following things. If an example exists, no justification is needed; otherwise, explain why no example exists.

a) A $3 \times 3$ matrix $A$ such that $\text{Col}(A) = \text{Nul}(A)$.
   
   No such matrix exists because $\dim \text{Col}(A) + \dim \text{Nul}(A) = 3$.

b) Row-equivalent $2 \times 2$ matrices $A$ and $B$ with $\text{Col}(A) \neq \text{Col}(B)$.
   
   There are many correct answers. For instance, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

c) A $3 \times 2$ matrix with full row rank.
   
   No such matrix exists, as a matrix with two columns cannot have three pivots.

d) A $2 \times 2$ matrix $A$ such that $Ax = 0$ is inconsistent.
   
   No such matrix exists: $Ax = 0$ always has the solution $x = 0$. 