Subspaces

Orientation: We're developing machinery to "almost solve" $Ax=b$

Today: give new names to everything we've been doing.

So far, to every matrix $A$ we have associated two spans:

1. the span of the columns/ all $b$ such that $Ax=b$
2. the solution set of $Ax=0$

The first arises naturally as a span/ it is already in parametric form. The second required work (elimination) to write as a span - it is a solution set, so it is in implicit form.

The notion of subspaces puts both on the same footing. This formalizes what we mean by "linear space containing 0".

Fast-forward:

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Subspaces are spans and Spans are subspaces.
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Why the new vocabulary word?
When you say "span" you have a spanning set of vectors in mind (parametric form). This is not the case for the solutions of $Ax=0$. **
Subspaces allow us to discuss spans without computing a spanning set. Subspace = Span \{ ?? \}

They also give a criterion for a subset to be a span.

Def: A subset of \( \mathbb{R}^n \) is any collection of points.

Eg: (a) \( \{(x,y): x^2+y^2=1\} \)  (b) \( \{(x,y): x,y \geq 0\} \)  (c) \( \{(x,y): xy=0\} \)

Def: A subspace is a subset \( V \) of \( \mathbb{R}^n \) satisfying:

1) [closed under +] If \( u, v \in V \) then \( u + v \in V \)

2) [closed under scalar \( x \)]
   If \( u \in V \) and \( c \in \mathbb{R} \) then \( cu \in V \)

3) [contains 0] \( 0 \in V \)

These conditions characterize linear spaces containing 0 among all subsets.

NB: If \( V \) is a subspace and \( v \in V \) then \( 0 = 0v \) is in \( V \) by (2), so (3) just means \( V \) is nonempty
Eg: In the subsets above:

(a) fails (1), (2), (3)
(b) fails (2): (1) $\in V$ but $-1 \cdot (1) \notin V$
(c) fails (1): (1), (0) $\in V$ but (1) $\notin V$

Here are two "trivial" examples of subspaces:

Eg: $\{0\}$ is a subspace

1. $0 + 0 = 0 \in \{0\}$ ✓
2. $c \cdot 0 = 0 \in \{0\}$ ✓
3. $0 \in \{0\}$ ✓

NB $\{0\} = \text{Span}\{0\}$; it is a span

Eg: $\mathbb{R}^n = \{\text{all vectors of size } n\}$ is a subspace

1. The sum of two vectors is a vector ✓
2. A scalar times a vector is a vector ✓
3. $0$ is a vector ✓

NB $\mathbb{R}^n = \text{Span}\{e_1, e_2, \ldots, e_n\}$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  ...  $e_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
The defining condition tells you if \((x, y, z)\) is in \(V\) or not.

(1) We have to show that if \(u = (x_1, y_1, z_1)\in V\) and \(v = (x_2, y_2, z_2)\in V\) then their sum is in \(V\).

\[
\text{Know: } x_1 + y_1 = z_1, \quad x_2 + y_2 = z_2
\]

defining conditions for \(u \& v\)

\[
u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)
\]

\[
\text{Want: } (x_1 + x_2) + (y_1 + y_2) = (z_1 + z_2) \quad \checkmark
\]

defining condition for \(u + v\).

Since \(u + v\) satisfies the defining condition, \(u + v \in V\).

(2) We have to show that if \((x, y, z)\in V\) and \(c \in \mathbb{R}\) then \(c(x, y, z) = (cx, cy, cz)\in V\).

\[
\text{Know: } x + ty = z \quad \text{Want: } cx + cy = cz \quad \checkmark
\]

Since \(cu\) satisfies the defining condition, \(cu \in V\).

(3) Is \((0, 0)\in V\)? Does it satisfy the defining condition?

\[
0 + 0 = 0 \quad \checkmark
\]

Since \(V\) satisfies the 3 criteria, it is a subspace.
NB: This means $V$ is a span!

How to find a spanning set? More on this later.

In order to show that a subset is not a subspace, you just have to produce one counterexample to one of the axioms.

**Eg:** $V = \{ (x,y) : x \geq 0, y \geq 0 \}$

$(2) \text{ is false: } (1,1) \in V \quad (1 \geq 0, 1 \geq 0)$

but $(-1)(1,1) \notin V \quad (-1 \not\geq 0, -1 \not\geq 0)$

In practice you will rarely check that a subset is a subspace by verifying the axioms — but you’ll show it’s not a subspace by finding a counterexample.

**Fact:** A span is a subspace

**Proof:** Let $V = \text{Span} \{ v_1, \ldots, v_n \}$.

Here the defining condition for a vector to be in $V$ is that it is a linear combination of $v_1, \ldots, v_n$. 
(1) We need to show that if
\[ c_1v_1 + \cdots + c_nv_n \in V \quad \& \quad d_1v_1 + \cdots + d_nv_n \in V \]
then their sum is in \( V \): the sum of two linear combos of \( v_1, \ldots, v_n \) is a linear combo.
\[
(c_1v_1 + \cdots + c_nv_n) + (d_1v_1 + \cdots + d_nv_n) \\
= (c_1+d_1)v_1 + \cdots + (c_n+d_n)v_n \in V
\]  

(2) We need to show that if \( c_1v_1 + \cdots + c_nv_n \in V \) and \( d \in \mathbb{R} \) then the product is in \( V \).
\[
d(c_1v_1 + \cdots + c_nv_n) = (dc_1)v_1 + \cdots + (dc_n)v_n \in V
\]

(3) Every span contains \( 0 \):
\[ 0 = 0v_1 + \cdots + 0v_n \]

Conversely, suppose \( V \) is a subspace.
If \( v_1, \ldots, v_n \in V \) and \( c_1, \ldots, c_n \in \mathbb{R} \) then:
\[
c_1v_1, \ldots, c_nv_n \in V \quad \text{by (2)} \\
c_1v_1 + c_2v_2 \in V \quad \text{by (1)} \\
(c_1v_1 + c_2v_2) + c_3v_3 \in V \quad \text{by (1)} \\
\vdots \\
c_1v_1 + \cdots + c_nv_n \in V
\]
so \( \text{Span \{v_1, \ldots, v_n\}} \) is contained in \( V \).

Choose enough \( v_i \)'s to fill up \( V \), and you get:
Def: The column space of a matrix $A$ is the span of its columns.

Notation: $\text{Col}(A) = \text{Span}\text{ cols of } A$

This is a subspace of $\mathbb{R}^m$ $m = \#\text{rows}$ (each column has $m$ entries).

$\sim$ column picture.

Since a column space is a span & a span is a subspace, a column space is a subspace.

Eg: $\text{Col}\begin{bmatrix} 1 & 4 & 7 \\ 3 & 6 & 9 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

Spans & Col spaces are interchangeable:

Eg: $\text{Span}\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}^3 = \text{Col}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

NB: $\text{Col}(A) = \{ Ax : x \in \mathbb{R}^n \}$

because "Ax" is just a linear combination of the cols of $A$. 
Translation of the column picture criterion for consistency:

\[ Ax = b \text{ is consistent } \iff b \in \text{Col}(A) \]

"b can be written as \( Ax \iff b \in \text{Col}(A) \)"

**Def:** The null space of a matrix \( A \) is the solution set of \( Ax = 0 \).

**Notation:** \( \text{Nul}(A) = \{ x \in \mathbb{R}^n : Ax = 0 \} \)

This is a subspace of \( \mathbb{R}^n \) \( n = \# \text{columns} \)
\( (n = \# \text{variables} \text{ and } \text{Nul}(A) \text{ is a solution set}) \)

\( \rightarrow \) row picture

**Fact:** \( \text{Nul}(A) \) is a subspace

Of course we also know \( \text{Nul}(A) \) is a span, but we can verify this directly.

**Proof:** The defining condition for \( \forall \in \text{Nul}(A) \) is that \( Av = 0 \).

(1) Say \( u, v \in \text{Nul}(A) \). Is \( u + v \in \text{Nul}(A) \)?
\[ A(u + v) = Au + Av = 0 + 0 = 0 \]
(2) Say \( u \in \text{Null}(A) \) and \( c \in \mathbb{R} \). Is \( cu \in \text{Null}(A) \)?

\[
A(cu) = c(Au) = c \cdot 0 = 0 \checkmark
\]

(3) Is \( 0 \in \text{Null}(A) \)?

\[
A0 = 0 \checkmark
\]

This is an example of a subspace that does not come with a spanning set!

\( \Rightarrow \) It's much more natural to consider it as a subspace when reasoning about it.

How to produce a spanning set for a null space?

\[
\text{null}(A)
\]

\text{Parametric vector form}

\( \text{(Gauss-Jordan eliminations)} \)

\text{Work}

\[
\text{Span } \{ \ldots \}
\]

\text{Eg: Write } \text{null}(A) \text{ as a span for } A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}

This means solving \( Ax = 0 \) (homogeneous equation).
\[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

parametric form
\[
\begin{align*}
x_1 &= -2x_2 + x_4 \\
x_2 &= x_2 \\
x_3 &= -x_4 \\
x_4 &= x_4
\end{align*}
\]

rref
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\xrightarrow{\text{rref}}
\begin{bmatrix}
\frac{-2}{1} \\
0 \\
1 \\
1
\end{bmatrix}
\]

\[
\implies \text{Null}(A) = \text{Span} \left\{ \begin{pmatrix}-2 \\ 0 \\ 1 \\ 1\end{pmatrix}, \begin{pmatrix}1 \\ 1 \\ 0 \\ 1\end{pmatrix} \right\}
\]

NB: Any two non-collinear vectors span a plane, so \( \text{Null}(A) \) will have many different spanning sets.

eg \( \text{Null}(A) = \text{Span} \left\{ \begin{pmatrix}-1 \\ 0 \\ 0 \\ 0\end{pmatrix}, \begin{pmatrix}1 \\ 0 \\ 0 \\ 1\end{pmatrix} \right\} \)

More on this later.

NB: Likewise for the column space: eg

\( \text{Col} \left( \begin{pmatrix}1 \\ 0 \\ 0 \\ 0\end{pmatrix} \right) = \text{Col} \left( \begin{pmatrix}1 \\ 1 \\ 0 \\ 0\end{pmatrix} \right) = \text{Col} \left( \begin{pmatrix}1 \\ 0 \\ 0 \\ 1\end{pmatrix} \right) = (xy\text{-plane}) \)
Implicit vs Parametric form:

- Col(A) is a span:
  \[ \text{Col} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 6 & 8 \end{pmatrix} = \text{the form} \]

  \[ \Rightarrow \text{parametric form} \]

- Nul(A) is a solution set:
  \[ \text{Nul} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix} \]

  \[ = \{ (x_1, x_2, x_3, x_4): x_1 + 2x_2 + 2x_3 + x_4 = 0, 2x_1 + 4x_2 + x_3 - x_4 = 0 \} \]

  \[ \Rightarrow \text{implicit form} \]

In practice you will (almost) always write a subspace as a column space/span or a null space. Which one?

- parameters? \[ \Rightarrow \text{Col}(A) / \text{Span} \]
- equations? \[ \Rightarrow \text{Nul}(A) \]

Once you're done this, you can ask a computer to do computations on it!
This is defined by the equation $xty=z$.

rewrite: $x+y-z=0$

$\Rightarrow V = \text{Nul} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

This is described by parameters. Rewrite:

$\begin{bmatrix} 3a+b \\ a-b \\ b \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow V = \text{Span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} = \text{Col} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

This is also how you should verify that a subset is a subspace.

Of course, if $V$ is not a subspace then you can't write it as $\text{Col}(A)$ or $\text{Nul}(A)$. In this case you should check that it fails one of the axioms.

Is $V = \{ (x,y,z) : xty = z+1 \}$ a subspace?

No, (P3) fails: $0+0 \neq 0+1$, so $0 \not\in V$. 