Geometry of Vectors

Recall: A vector in \( \mathbb{R}^n \) is a list of \( n \) numbers:

\[ v = (x_1, \ldots, x_n) \in \mathbb{R}^n \]

We can draw a vector as a point in Euclidean space:

\[ (x_1, x_2) = (x\text{-coordinate, } y\text{-coordinate}) \]

We will often consider a vector as an arrow or displacement: measures the difference between two points.

\[ (x_1) = (x\text{-displacement}) \]
\[ (x_2) = (y\text{-displacement}) \]

\[ \text{NB the tail of the vector can be anywhere, but by default vectors start at 0} \]

How do algebraic operations behave geometrically? We’ll describe in terms of arrows.
Scalar Multiplication:
- the length of \( cv \) is \( |c| \times \) the length of \( v \)
- the direction of \( cv \) is:
  - the same as \( v \) if \( c > 0 \)
  - the opposite from \( v \) if \( c < 0 \)

Eg: \( v = (2, 3) \)

\[
2v = (4, 6) \\
-v = (-2, -3)
\]

Vector Addition:
This just adds the displacements.

Parallelogram Law: to draw \( v + w \),
draw the tail of \( v \) at the head of \( w \)
(or vice-versa); the head of \( v \) is at \( v + w \).

Eg: \( v = (2, 3) \) \\
\( w = (1, 2) \) \\
\( v + w = (3, 5) \)
Vector Subtraction: \( \omega + (v - \omega) = v \)
So \( v - \omega \) starts at the head of \( \omega \) & ends at the head of \( v \).

\[
\begin{align*}
\text{Eg:} & \quad v = (\frac{3}{2}) \\
\omega = (\frac{1}{2}) \\
\therefore v - \omega = (-1)
\end{align*}
\]

Linear Combinations:
First scale, then add.

\[
\begin{align*}
\text{Eg:} & \quad v = (\frac{3}{2}) \\
\omega = (\frac{1}{2}) \\
\therefore 2v + \omega & = (\frac{4}{2}) \\
\therefore 2v - \omega & = (\frac{1}{2})
\end{align*}
\]

This is like giving directions: "To get to 
\(-1.5v + 0.5\omega\), first go \(1.5\times \text{length of } v\) in the 
opposite \(v\)-direction, then go \(0.5\times \text{length of } \omega\) in the 
\(w\)-direction."
Spans: Look out for two subtle concepts below.

Recall: the notion of "all linear combinations of some set of vectors" came up twice last time:

- $Ax=b$ is consistent if $b \in \text{Span}(\text{all linear combinations of the columns of } A)$.
- If so, the solution set of $Ax=b$ is $(\text{particular solution}) + (\text{all linear combinations of some vectors})$.

Def: The span of a list of vectors is the set of all linear combinations of those vectors:

$$\text{Span}\{v_1, v_2, \ldots, v_n\} = \left\{ c_1v_1 + c_2v_2 + \cdots + c_nv_n : c_1, \ldots, c_n \in \mathbb{R} \right\}$$

This is set-builder notation.

Translation of the above:

1. $Ax=b$ is consistent $\iff b \in \text{Span}\{\text{columns of } A\}$.
2. If so, the solution set of $Ax=b$ is $(\text{particular solution}) + \text{Span}\{\text{some vectors}\}$. 
Column Picture Criterion for Consistency (again)

\[ Ax = b \] is consistent (has at least one solution)

\( \Downarrow \)

be \( \text{Span} \) \{columns of } A \}

What do spans look like?

It's the smallest "linear space" (line, plane, etc.) containing all your vectors & the origin.

Eg: \( \text{Span} \{ v \} = \{ cv: c \in \mathbb{R} \} \)

- If \( v \neq 0 \) get the line thru \( \mathbf{0} \) & \( v \)
- \( \text{Span} \{ \mathbf{0} \} = \{ c \cdot \mathbf{0}: c \in \mathbb{R} \} = \{ \mathbf{0} \} \)
  = the set containing only \( \mathbf{0} \) \( [\text{demo}] \)

Eg: \( \text{Span} \{ v, w \} = \{ cv + dw: c, d \in \mathbb{R} \} \)

- If \( v, w \) are not collinear, get the plane defined by \( \mathbf{0}, v, \) and \( w \)
- If \( v, w \) are collinear and nonzero, get the line thru \( v, w, \) and \( \mathbf{0} \)
- If \( v = w = \mathbf{0} \) get \( \mathbb{R}^3 \) \( [\text{demo}] \)
If \( u, v, w \) are not coplanar, get space

If \( u, v, w \) are coplanar but not collinear, get the plane containing them.

If \( u, v, w \) are collinear & not all zero, get the line through \( u, v, w, \) and 0.

If \( u = v = w = 0 \) get \( \{0\} \) 

\[ \text{Eg: } \text{Span} \{u, v, w\} = \text{Span} \{0, 0, 0\} \] (by convention)

Warning: Be careful to distinguish these sets:

- \( \emptyset \): the empty set has no vectors in it at all (e.g. the solution set of an inconsistent system)
- \( \{0\} \): the point contains (only) the zero vector

The difference is: \( \{0\} \) contains 0; \( \emptyset \) does not.

Likewise,

- \( \{u_1, \ldots, u_n\} \): a set with \( n \) vectors in it
- \( \text{Span} \{u_1, \ldots, u_n\} \): a linear space; it contains infinitely many vectors (unless \( u_1 = \cdots = u_n = 0 \))

eg. a line
The span construction allows you to parametrically describe a linear space (infinite set) using a finite amount of data.

Now you can do computations!

**NB:** Every span contains the zero vector!

\[ O = 0 \cdot v_1 + 0 \cdot v_2 + \ldots + 0 \cdot v_n \]

So e.g., this line is not a span:

\[ \text{Eg: this line is not a span! It does not contain } O. \]

**Q:** Is \( \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \) in \( \text{Span}\{ \begin{bmatrix} 1 \\ 2 \\ 6 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \} \)?

In other words, does

\[ x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \]

have a solution?

Let’s solve this vector equation:

\[
\begin{bmatrix} 1 & -1 & 16 \\ 2 & -2 & 6 \\ 6 & -1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = -1, x_2 = -9
\]

Answer: yes, \( \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \in \text{Span}\{ \begin{bmatrix} 1 \\ 2 \\ 6 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \} \)
This example is just the "⇒" of the statement: 

\[ Ax=b \text{ is consistent} \iff b \in \text{Span } \text{cols of } A \]

Column Picture Criterion for Consistency:

1. \[
\begin{pmatrix}
\frac{1}{2} & -2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =
\begin{pmatrix} 8 \\ 3 \end{pmatrix}
\]
   is consistent because
   \[
   \begin{pmatrix} 8 \\ 3 \end{pmatrix} \in \text{Span } \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}
   \]
   [demo]
   
   **row picture:**
   \[
   \begin{align*}
x_1 - x_2 &= 8 \\
2x_1 - 2x_2 &= 16 \\
6x_1 - x_2 &= 3
   \end{align*}
   \]

2. \[
\begin{pmatrix}
\frac{1}{2} & -2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =
\begin{pmatrix} 7 \\ -1 \end{pmatrix}
\]
   is inconsistent because
   \[
   \begin{pmatrix} 7 \\ -1 \end{pmatrix} \notin \text{Span } \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}
   \]
   [demo]
   
   **row picture:**
   \[
   \begin{align*}
x_1 - x_2 &= 7 \\
2x_1 - 2x_2 &= 1 \\
6x_1 - x_2 &= -1
   \end{align*}
   \]
Homogeneous Equations

If the solution set of $Ax=b$ is a span

$\Rightarrow 0$ is a solution (every span contains $0$)

$\Rightarrow A0=b \implies b=0$

Let's study this case.

Def: $Ax=b$ is called homogeneous if $b=0$.

Eg: $x_1+2x_2+2x_3+x_4=0$
$2x_1+4x_2+x_3-x_4=0$

NB: A homogeneous equation is always consistent since $0$ is a solution: $A \cdot 0=0$

Def: The trivial solution of a homogeneous equation $Ax=0$ is the zero vector.

Eg: Let's solve the homogeneous system

$x_1+2x_2+2x_3+x_4=0 \implies \begin{bmatrix} 1 & 2 & 2 & 1 & 0 \end{bmatrix}$

$2x_1+4x_2+x_3-x_4=0$

$R_2\leftarrow 2R_1 \begin{bmatrix} 1 & 2 & 2 & 1 & 0 \end{bmatrix}$

$R_2 \rightarrow \begin{bmatrix} 0 & 0 & -3 & -3 & 0 \end{bmatrix}$

$R_2 \rightarrow -3 \begin{bmatrix} 1 & 2 & 2 & 1 & 0 \end{bmatrix}$

$R_2 \rightarrow 2R_2 \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \end{bmatrix}$
\[
\begin{align*}
\text{PF:} & \quad x_1 = -2x_2 + x_4 \\
& \quad x_2 = x_2 \\
& \quad x_3 = -x_4 \\
& \quad x_4 = x_4
\end{align*}
\]

\[
\text{PVF:} \quad x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

**Observations:**

1. The augmented column is always zero.
   When solving homogeneous equations, you don’t need to write the augmented column.

\[
\begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 \end{pmatrix}
\]

2. The particular solution is the zero vector.
3. The solution set is
\[
\text{Span} \begin{Bmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \end{Bmatrix}
\]

**Fact:** The PVF of a homogeneous system always has particular solution = 0. The solution set is the span of the other vectors you’ve produced.
Inhomogeneous Equations

Def: $Ax=b$ is called inhomogeneous if $b \neq 0$.

What's the difference from homogeneous equations?

NB: It can be inconsistent!

Let's solve the inhomogeneous & homogeneous versions:

Eg: inhomogeneous

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Homogeneous

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(augmented) matrix

Let's solve the inhomogeneous & homogeneous versions:

Eg: inhomogeneous

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Homogeneous

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

RREF same

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

PYF

$$x = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + z \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

Solution set same

$$\text{Span} \left\{ \begin{pmatrix} -5 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -5 \\ 1 \end{pmatrix} \right\}$$
The only difference is the particular solution! Otherwise they're parallel lines.

Facts:

1. The solution set of $Ax = 0$ is a span.

2. The solution set of $Ax = b$ is not a span.
   - For $b \neq 0$: it is a translate of the solution set of $Ax = 0$ by a particular solution. (Or it is empty.)

\[
\text{(solutions of } Ax = 0) = \text{(zero)} + \text{Span } \{ \text{vectors from PUF} \}
\]

\[
\text{(solutions of } Ax = b) = \text{(particular solution)} + \text{Span } \{ \text{vectors from PUF} \}
\]
In fact, to get the solutions of $Ax = b$ you can translate the solutions of $Ax = 0$ by any single solution of $Ax = b$.

Say $p$ is some solution of $Ax = b$, so $Ap = b$. Then $Ax = 0 \iff Ap + Ax = b \iff A(p + x) = b$ vectors of the form $p + (\text{soln of } Ax = 0)$

NB: Expressing a solution set as a (translate of a) span means writing it in parametric form:

$x \in \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \begin{Bmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{Bmatrix}$

$\iff x = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

So think:

\[ \text{Spans} \quad \equiv \quad \text{Parametric form} \]
We now know:

1. \((\text{All solutions of } Ax=b) = (\text{some solution of } Ax=b) + (\text{All solutions of } Ax=0)\)

or is empty. In particular, all nonempty solution sets are parallel and look the same.

2. \(Ax=b\) is consistent if and only if \(b\) is in the span of the columns of \(A\).

We can draw these both at the same time:

In this picture, we think of \(A\) as a function:

- \(x \in \mathbb{R}^n\) is the input (row picture)
- \(Ax \in \mathbb{R}^m\) is the output (column picture)

Solving \(Ax=b\) means finding all inputs with output \(=b\).
The solution set lives in the row picture!
The $b$-vectors live in the column picture!
The columns all live in the column picture!
That's how you keep them straight.