Complex Eigenvalues

Some matrices have no (real) eigenvalues. But every matrix has a complex eigenvalue: any polynomial \( p(x) \) has a complex zero.

**Eg:** \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) (CCW rotation by 90°)

\[ p(x) = x^2 + 1 = (x+i)(x-i) \]

Diagonalization still works great even if the eigenvalues are not real.

→ Still can solve difference equations & ODEs

→ Still get real-number answers

So we can apply diagonalization techniques to more matrices if we allow complex eigenvalues.

**Fact:** The complex eigenvalues \& eigenvectors of a real matrix come in complex conjugate pairs:

\[ A \mathbf{v} = \lambda \mathbf{v} \iff A \overline{\mathbf{v}} = \lambda \overline{\mathbf{v}} \]

here \( \mathbf{v} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \) \( \Rightarrow \overline{\mathbf{v}} = \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix} \)
Eg: Solve the difference equation
\[ v_{k+1} = A v_k \quad A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \quad v_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \]

No complex #s in the statement!

(1) Diagonalize:
\[ p(\lambda) = \lambda^2 + 3\lambda + 3 \Rightarrow \lambda = \frac{1}{2} \left( -3 \pm \sqrt{9 - 12} \right) \]
\[ \Rightarrow \lambda = \frac{1}{2} (-3 + i\sqrt{3}), \quad \bar{\lambda} = \frac{1}{2} (-3 - i\sqrt{3}) \]

Find eigenvectors using the 2x2 trick
\[ \omega = \begin{pmatrix} -b \\ a - \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \quad \bar{\omega} = \begin{pmatrix} 1 \\ -\bar{\lambda} \end{pmatrix} \]

eigenvector for \( \lambda \)  \quad  \text{eigenvector for} \  \bar{\lambda} \]

Check: \[ A \omega = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 3 + 3\lambda \end{pmatrix} \]

Wait, is this equal to \( \lambda \omega \)?
\[ \lambda \omega = \lambda \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \lambda^2 \end{pmatrix} \]
\[ = \begin{pmatrix} \lambda \end{pmatrix} \]
\[ \Rightarrow -\lambda^2 = 3 + 3\lambda \quad \text{because} \]
\[ \lambda^2 + 3\lambda + 3 = p(\lambda) = 0 \quad \checkmark \]

So \( \{\omega, \bar{\omega}\} \) is an eigenbasis
(different eigenvalues \( \Rightarrow \) LI)
(2) Expand the initial state in our eigenbasis:

We need to solve \((3) = V_0 = x, \omega + x_2 \bar{\omega}\). 
\[
\begin{pmatrix}
1 & 1 \\
-\lambda & -\lambda \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 \\
1 - \lambda \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
\Rightarrow (2 \lambda^2 - \lambda + 3) = \lambda^2 - 3x_1 - 2x_2 = \lambda x_1^2 - x_2^2 \\
\Rightarrow \begin{cases}
x_1 + x_2 = 2 \\
(3 \lambda^2) x_2 = 3 \lambda \\
\end{cases} \Rightarrow x_2 = 1 \rightarrow x_1 = 1
\]
(back-substitution)

So \( V_0 = w_1 + w_2 \)

Solution: \( A^k V_0 = \lambda^k w + \bar{\lambda}^k \bar{\omega} \)

So far it's exactly the same as for real eigenvalues! 
... but we wanted a solution involving only real \#s.

Thankfully, \( \lambda^k w \) and \( \bar{\lambda}^k \bar{\omega} \) are complex conjugates, so

\[
A^k V_0 = \lambda^k w + \bar{\lambda}^k \bar{\omega} = 2 \text{Re}[\lambda^k w]
\]
\[
= 2 \text{Re}[\lambda^k (\frac{1}{\lambda})] = 2 \text{Re}(\frac{\lambda^k}{\lambda^{k+1}})
\]

Recall: Multiplication of complex numbers is much easier in polar form.
\[ \lambda = \frac{1}{2} (-3 + \sqrt{3}) = r e^{i \theta} \]
\[ r = \frac{1}{2} \sqrt{9 + 3} = \frac{1}{2} \sqrt{4 - 3} = \frac{1}{2} \sqrt{3} \]
\[ \theta = 150^\circ = \frac{5 \pi}{6} \]

**Euler's Formula**

So, \[ x^k = r^k e^{i k \frac{5 \pi}{6}} \]

\[ \Rightarrow \text{Re}(x^k) = (\sqrt{3})^k \cos \left( \frac{5 k \pi}{6} \right) \]

\[ \Rightarrow v_k = 2 \begin{pmatrix} \sqrt{3}^k \cos \left( \frac{5 k \pi}{6} \right) \\ -\sqrt{3}^{k+1} \cos \left( \frac{5 (k+1) \pi}{6} \right) \end{pmatrix} \]  

[demo]

The answer involves only real numbers (and cosines—weird!) but we needed complex numbers to get it!

**Difference Equations with Complex Eigenvalues:**

To solve \( v_{k+1} = A v_k \):

1. **Diagonalize** \( A \) and expand \( v_0 \) in an eigenbasis, as before. Complex numbers are OK.

   \[ \Rightarrow \text{Remember } A v = \lambda v \iff A \bar{v} = \bar{\lambda} \bar{v} \]

2. **Group complex conjugate terms:**

   \[ x^k x \omega + \bar{x}^k \bar{x} \bar{\omega} = 2 \text{Re} (x^k x \omega) \]
(4) Write \( \lambda \) in polar form:

\[ \lambda = r e^{i \theta} \Rightarrow \lambda^k = r^k e^{i k \theta} = r^k (\cos k \theta + i \sin k \theta) \]

Multiply this by \( x \) and the coordinates of \( w \) and take the real part.

We get an answer with sines & cosines (but no \( i \)'s).

\[ \lambda = 1 + i \quad x = 3 - 2i \quad \omega = (2i) \]

\[ \lambda = \sqrt{2} e^{i \pi/4} \Rightarrow \lambda^k = (\sqrt{2})^k e^{i k \pi/4} \]

\[ = 2^{k/2} (\cos \frac{k \pi}{4} + i \sin \frac{k \pi}{4}) \]

\[ \Rightarrow \lambda^k \text{Re} = 2^{k/2} (\cos \frac{k \pi}{4} + i \sin \frac{k \pi}{4}) (3 - 2i)(2i) \]

\[ = 2^{k/2} \left[ 3 \cos \frac{k \pi}{4} + 2 \sin \frac{k \pi}{4} + i \left( 3 \sin \frac{k \pi}{4} - 2 \cos \frac{k \pi}{4} \right) \right] (2i) \]

\[ = 2^{k/2} \left( 3 \cos \frac{k \pi}{4} + 2 \sin \frac{k \pi}{4} + i \left( 3 \sin \frac{k \pi}{4} - 2 \cos \frac{k \pi}{4} \right) \right) \]

\[ = 2^{k/2} \left( -6 \sin \frac{k \pi}{4} + 4 \cos \frac{k \pi}{4} + i \left( 6 \cos \frac{k \pi}{4} + 4 \sin \frac{k \pi}{4} \right) \right) \]

\[ \Rightarrow 2 \text{Re} [\lambda^k x \omega] = 2 \cdot 2^{k/2} \left( -6 \sin \frac{k \pi}{4} + 4 \cos \frac{k \pi}{4} \right) \]
Algebraic & Geometric Multiplicity

Last we will discuss a criterion for diagonalizability.

We like diagonalizable matrices because we can solve difference equations.

Recall: If $\lambda$ is a root of a polynomial $p(x)$, its multiplicity $m$ is the largest power of $(x-\lambda)$ dividing $p$: $p(x) = (x-\lambda)^m \cdot \text{(other factors)}$

Eg: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = - (\lambda - 2)^2 (\lambda + 1)^1$

$\lambda = 2$ has multiplicity 2; $\lambda = -1$ has multiplicity 1

Def: Let $A$ be an $n \times n$ matrix with eigenvalue $\lambda$.

1. The **algebraic multiplicity (AM)** of $\lambda$ is its multiplicity as a root of the characteristic polynomial $p(\lambda)$.

2. The **geometric multiplicity (GM)** of $\lambda$ is the dimension of the $\lambda$-eigenspace:

   $GM(\lambda) = \dim \text{Null}(A - \lambda I_n)$

   = number of free variables in $A - \lambda I_n$

   = number of linearly independent $\lambda$-eigenvectors
Eg: \[ A = \begin{pmatrix} -7 & 3 & 5 \\ -6 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix} \quad p(\lambda) = - (\lambda - 2)^2 (\lambda - 1) \]

So the eigenvalues are 1 & 2.

- \( \lambda = 1 \): \( \text{AM} = 1 \).
  
  \[ \text{Null}(A - 1I_3) = \text{Span} \{ (1) \} \]
  
  \( \rightarrow \) this is a line: \( \text{GM} = 1 \)

- \( \lambda = 2 \): \( \text{AM} = 2 \)
  
  \[ \text{Null}(A - 2I_3) = \text{Span} \{ (\frac{3}{2}, \frac{1}{2}, 0) \} \]
  
  \( \rightarrow \) this is a line: \( \text{GM} = 1 \)

This matrix is not diagonalizable:

- only two linearly independent eigenvectors.

[demo]
Eg: \[ B = \begin{pmatrix} -4 & 3 & 3 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix} \] \[ p(\lambda) = -(\lambda-2)^2(\lambda-1) \]

So the eigenvalues are 1 & 2.

- \( \lambda = 1 \): \( AM = 1 \)
  \[ \text{Null}(B - 1I_3) = \text{Span} \{ (1) \} \]
  \[ \rightarrow \text{this is a line: } GM = 1 \]
  \[ \text{AM} = \text{GM} \]

- \( \lambda = 2 \): \( AM = 2 \)
  \[ \text{Null}(B - 2I_3) = \text{Span} \{ (\frac{3}{2}), (\frac{1}{2}) \} \]
  \[ \rightarrow \text{this is a plane: } GM = 2 \]
  \[ \text{AM} = \text{GM} \]

This matrix is diagonalizable: an eigenbasis is \[ \{ (1), (\frac{3}{2}), (\frac{1}{2}) \} \]

Both matrices have only 2 eigenvalues.

The difference is that \( B \) had \( AM = GM = 2 \) LI 2-eigenvectors and \( A \) had one.
Thm (AM ≥ GM): For any eigenvalue $\lambda$ of $A$,
(algebraic multiplicity of $\lambda$)
\[ \geq \text{(geometric multiplicity of } \lambda) \geq 1 \]

For a proof, see the supplement.

NB: GM ≥ 1 just says every eigenvalue has an eigenvector — the eigenspace can't be $\emptyset$, so its dimension is ≥ 1.

Upshot: if $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$ then
- the 1-eigenspace is necessarily a line:
  \[ AM = 1 \geq GM \geq 1 \]
- the 2-eigenspace is a line or a plane:
  \[ AM = 2 \geq GM \geq 1 \]
- the matrix is diagonalizable if $GM(2) = 2$:
  then you have $1 + 2 = 3$ LI eigenvectors.
Thm (AM/GM Criterion for Diagonalizability):

Let $A$ be an $n \times n$ matrix.

- $A$ is diagonalizable over the complex numbers if $AM(\lambda) = GM(\lambda)$ for every eigenvalue $\lambda$.
- $A$ is diagonalizable over the real numbers if $AM(\lambda) = GM(\lambda)$ for every eigenvalue $\lambda$ and $A$ has no complex eigenvalues.

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ is not diagonalizable because $AM(2) = 2 \neq 1 = GM(2)$.

Corollary: If $A$ has $n$ different eigenvalues then $A$ is diagonalizable.

Proof: If $A$ has $n$ different eigenvalues then

$n = AM(\lambda_1) + \cdots + AM(\lambda_n) \implies AM(\lambda_i) = 1$

$1 = AM(\lambda_i) = GM(\lambda_i) \geq 1 \implies AM(\lambda_i) = GM(\lambda_i) = 1$

Eg: A $2 \times 2$ real matrix with a complex eigenvalue $\lambda$ is diagonalizable (over $\mathbb{C}$): it has 2 eigenvalues $\lambda$ and $\bar{\lambda}$. 
Proof of the Theorem:

First note that

\[ p(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_r)^{m_r} \]

factors into linear factors (over \(\mathbb{C}\)), where \(m_i = AM(\lambda_i)\). Hence

\[ AM(\lambda_1) + \ldots + AM(\lambda_r) = n \quad \text{(sum of the } AM's \text{ is } n) \]

If \(A\) is diagonalizable then it has \(n\) LI eigenvectors, so

\[ n = GM(\lambda_1) + \ldots + GM(\lambda_r) \]

\[ AM(\lambda_1) + \ldots + AM(\lambda_r) = n \]

This forces \(AM(\lambda_i) = GM(\lambda_i)\). Conversely, if each \(AM(\lambda_i) = GM(\lambda_i)\) then

\[ n = GM(\lambda_1) + \ldots + GM(\lambda_n) \]

so when you combine eigenspace bases you get \(n\) LI eigenvectors.