Linear Independence of Eigenvalues
Recall from last time: to diagonalize an $n x_{n}$ matrix $A$ :
(1) Compute $p(\lambda)=\operatorname{det}\left(A-\lambda I_{n}\right)$
(2) Sole $p(\lambda)=0$ to find the eigenvalues
(3) Find a basis for each erigengace
(4) Combine all these bases.
8. If yon end up with $n$ vectors, they're LI

- Otherwise $A_{i s}$ not diagonalizable

In \& we reed to justify why the eigenvectors are LI.
Fact: If $w_{1} \ldots, u_{p}$ are eigenvectors of $A$ with different eigenvalues then $\left\{\omega_{y}, \ldots, \omega_{p}\right\}$ is LI.
Here's how the Fact implies \&. Suppose

- $\left\{_{\omega_{1}}, \omega_{2}\right\}_{\text {is }}$ a basis for the $\lambda_{1}$-eigenspace
- $\left\{\omega_{3}\right\}$ is a basis for the $\lambda_{2}$-eigenspace.

I claim $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$ B $L I$.
Suppose $x_{1} \omega_{1}+x_{2} \omega_{2}+x_{3} \omega_{3}=0$. We need $x_{1}=x_{2}=x_{3}=0$.

- $x_{1} w_{1}+x_{2} \omega_{2}$ is in the $\lambda_{1}$-eigenspace
- Since $\left(x_{1} \omega_{1}+x_{2} \omega_{2}\right)+x_{3} \omega_{3}=0$, the Fact implies $x_{1} \omega_{1}+x_{2} \omega_{2}=0$ and $x_{3} \omega_{3}=0$ (30 $\left.x_{3}=0\right)$
- Since $\left\{w_{1, w_{2}}\right\}$ is LI, this implies $x_{1}=x_{2}=0$

Proof of the Fact: Say $A_{\omega_{i}}=\lambda_{i} \omega_{i}$ and all of the $\lambda_{1, \ldots} \ldots$ ip are distinct. Suppose $\left\{\omega_{1}, \ldots, \omega_{p}\right\}$ is LD. Then for some is $\left\{w_{1}, \ldots w_{i}\right\}$ is LI but $\omega_{i+1} \in \operatorname{Spin}\left\{\omega_{i, \ldots}, \omega_{i}\right\}$, so

$$
\begin{aligned}
& \omega_{i+1}=x_{1} \omega_{1}+\cdots+x_{i} \omega_{i} \\
& \Rightarrow A \omega_{i+1}=A\left(x_{1} \omega_{1}+\cdots+x_{i} \omega_{i}\right) \\
& \Rightarrow \lambda_{i+1} \omega_{i+1}=\lambda, x_{1} \omega_{1}+\cdots+\lambda_{i} x_{i} \omega_{i}
\end{aligned}
$$

If $\lambda_{i t 1}=0$ then $\lambda_{1} x_{1} \omega_{1}+\cdots+\lambda_{i} x_{i} \omega_{i}=0 \xrightarrow[L I]{\left.\hat{\omega}_{0} \ldots \omega_{i}\right\}}$ $x_{1}=\cdots=x_{i}=0 \quad$ (because $\lambda_{1}, \ldots, \lambda_{i} \neq 0$ ), so $\omega_{i+c}=0$, which can't happen because $w_{i+1}$ is an eigenvector. If $\lambda_{i+1} \neq 0$ then

$$
\omega_{i+1}=\frac{\lambda_{1}}{\lambda_{+1}} x_{1} \omega_{1}+\cdots+\frac{\lambda_{i}}{\lambda_{i+1}} x_{i} \omega_{i}
$$

Subtract $\omega_{i+1}=x_{1} \omega_{1}+\cdots+\quad x_{i} \omega_{i}$

$$
\leadsto 0=\left(\frac{\lambda_{1}}{\lambda_{i+1}}-1\right) x_{1} \omega_{1}+\cdots+\left(\frac{\lambda_{i}}{\lambda_{i+1}}-1\right) x_{i} \omega_{i}
$$

But $\lambda_{j} \neq \lambda_{i+1}$ for $j \leqslant i$, so $\frac{\lambda_{j}}{\lambda_{i+1}}-1 \neq 0$

$$
\Rightarrow x_{i}=\cdots=x_{i}=0
$$

which is impossible, as before.

Consequence: If $A$ has $n$ (different) eigenvalues then $A$ is diagonalizable.
Indeed, if $\lambda_{y} \not \lambda_{n}$ are eigenvalues and

$$
A w_{1}=\lambda_{1} \omega_{1}, \ldots, A \omega_{n}=\lambda_{n} \omega_{n}
$$

Then $\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ is an eigentasisis by the Fact.
Well give a moe general criterion (AM/G-M) next the.

Matrix Form of Diagonalization
Ohm: $A$ is diagonalizable $\Longleftrightarrow$ there exists an invertible matrix $C$ and a diagonal matrix $D$ such that

$$
A=C D C^{-1}
$$

In this case the columns of $C$ form an eigenbaris \& the diagonal entries of $D$ are the corresponding eigenvalues.

$$
C=\left(\begin{array}{cc}
1 & 1 \\
\omega_{1} & \cdots \\
1 & \omega_{n} \\
1 & \\
C_{\text {sane and er }}
\end{array}\right)_{-3} \quad D=\left(\begin{array}{lll}
\lambda_{1} & & \\
& \ddots & \lambda_{n}
\end{array}\right) \quad A \omega_{i}=\lambda_{i} \omega_{i}
$$

Eg:

$$
\begin{aligned}
& A=\left(\begin{array}{ccc}
0 & 13 & 12 \\
1 / 4 & 0 & 0 \\
0 & 1 & 0 \\
1_{2} & 0
\end{array}\right) \Rightarrow A=C D C^{-1} \text { for } \\
& C=\left(\begin{array}{ccc}
32 & 2 & 18 \\
4 & -1 & -3 \\
1 & 1 & 1
\end{array}\right) \quad D=\left(\begin{array}{ccc}
2 & 0 & 0 \\
0 & -1 / 2 & 0 \\
0 & 0 & -3 / 2
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
\text { Eg: } & A=\left(\begin{array}{ccc}
14 & -18 & -33 \\
-12 & 20 & 33 \\
12 & -18 & -1
\end{array}\right) \Rightarrow A=C D C^{-1} \text { for } \\
1 & w_{2} \\
\omega_{1} & \text { (1 )at the }) \\
& C=\left(\begin{array}{ccc}
3 & 11 & 1 \\
2 & 0 & -1 \\
0 & 4 & 1
\end{array}\right) \quad D=\left(\begin{array}{ccc}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{array}\right)
\end{aligned}
$$

Proof: $C\binom{x_{1}}{x_{n}}=x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}$

$$
\begin{aligned}
& \left(\dot{x}_{n}\right)=x_{1} \omega_{1}+\cdots+x_{n} \omega_{n} \\
& \Rightarrow C^{-1}\left(x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}\right)=\left(\begin{array}{l}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)
\end{aligned}
$$

Any vector has the form $v=x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}$, and two matrices ore equal if they act the same on every vector. So check:

$$
\begin{aligned}
C D C^{-1} r & =C D C^{-1}\left(x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}\right) \\
= & C\left(\begin{array}{ccc}
\lambda_{1} & \cdots & 0 \\
0 & \lambda_{n}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right) \\
& =\left(\begin{array}{cc}
1 & 1 \\
\omega_{1} & \cdots \\
1 & v_{n}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
x_{1} \\
\vdots \\
\lambda_{n}
\end{array}\right)=x_{n} \lambda_{1} \omega_{1}+\cdots+x_{n} \lambda_{n} \omega_{n} \\
& =A\left(\begin{array}{ll}
x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}
\end{array}\right)=A V
\end{aligned}
$$

$N B$ : If $A=C D C^{-1}$ then

$$
\begin{aligned}
A^{k}=\left(C D C^{-1}\right)^{k} & =\left(C C^{-1}\right)\left(C D C^{-1}\right) \ldots\left(C D C^{-1}\right) \\
& =C D^{k} C^{-1}=C\left(\begin{array}{ccc}
\lambda_{1}^{k} & 0 & 0 \\
0 & \lambda_{n}
\end{array}\right) C^{-1}
\end{aligned}
$$

This a a closed for expression for $A^{k}$ in terms of $k$ : much easier to compute!

$$
A^{k}=C D^{k} C^{-1} \leftarrow \text { thine matrix has } n^{2} \text { entries }
$$

Compare: $A^{k}\left(x_{1} \omega_{1}+\cdots+x_{n} \omega_{r}\right)=\lambda_{1}^{k} x_{1} \omega_{1}+\cdots+\lambda_{n}^{k} x_{n} \omega_{n}$ (vector foo of the same identity).
Eg: $A=\left(\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right)$ is diagonal:

$$
A e_{1}=2 e_{1} \quad A e_{2}=3 e_{2} \quad A e_{3}=4 e_{3}
$$

So $\left\{e_{1}, e_{2}, e_{3}\right\}$ a an eigenbasis as can take $C=I_{3}$, so the diagonalization is

$$
A=I_{3} A I_{3}
$$

Q: What if we take $e_{2}$ to be our first eigenvector?
$N B: A$ matrix is diagonal $\Leftrightarrow$ the unit coordinate vectors $e_{1} \ldots l_{n}$ are eigenvectors.

Geometry of Diagonalizable Matrices
When $A$ is diagonalizable, every vector can be written as a linear combination of eigenvectors, so multiplication by $A$ is reduced to scalar multiplication:

$$
A\left(x_{1} \omega_{1}+\cdots+x_{n} \omega_{n}\right)=\lambda_{1} x_{1} \omega_{1}+\cdots+\lambda_{n} x_{n} \omega_{n} .
$$

What does this mean geometrically?
$\rightarrow$ Expanding $m$ an eigenbasis and scalar multiplication can both be famulated geometrically!
NB: "Visualizing" a matrix means understanding how $x$ relates to $A x$; think of $A$ as $a$ function


Eg: $D=\left(\begin{array}{ll}2 & 0 \\ 0 & 1 / 2\end{array}\right)$ so $D\binom{x}{y}=\binom{2 x}{1 / 2 y}$

- scales the $x$-direction $b_{j} 2$
- Scales the $y$-direction by $1 / 2$


$$
\begin{gathered}
E g: A=\frac{1}{10}\left(\begin{array}{ll}
11 & 6 \\
9 & 14
\end{array}\right) \quad p(\lambda)=\lambda^{2}-\frac{5}{2} \lambda+1=(\lambda-2)\left(\lambda-\frac{1}{2}\right) \\
\lambda_{1}=2 \quad \omega_{1}=\binom{2}{3} \quad \lambda_{2}=\frac{1}{2} \quad \omega_{2}=\binom{-1}{1} \text { un } 1 \omega_{2} v_{2}=\lambda_{2}^{2},
\end{gathered}
$$

Expand in the eiganbersis
(thank in terns of $L\left(s\right.$ of $\left.u, v_{2}\right)$

$$
A\left(x_{1} \omega_{1}+x_{2} \omega_{2}\right)=2 x_{1} \omega_{1}+\frac{1}{2} x_{2} \omega_{2}
$$

- scales the $u$-direction $b_{1} 2$
- Scales the $\omega_{2}$-direction by $1_{2}$
[demo]


This is the vector for. $I_{n}$ matrix form

$$
A=C D C^{-1} \quad C=\left(\begin{array}{cc}
2 & -1 \\
3 & 1
\end{array}\right) \quad D=\left(\begin{array}{ll}
2 & 0 \\
0 & 1 / 2
\end{array}\right)
$$

Then $A r=C D C^{-1} V$
$="$ - first multiply $v$ by $C^{-1}$

- Then multiply by the diagonal matron $D$
- then multiply by $C$ again ${ }^{n}$

Note $C\binom{x_{1}}{x_{2}}=x_{1} \omega_{1}+x_{2} \omega_{2} \Longleftrightarrow C^{-1}\left(x_{1} \omega_{1}+x_{2} \omega_{2}\right)=\left(\begin{array}{l}x_{1}^{x_{2}}\end{array}\right)$

$E_{g}: D=\left(\begin{array}{ll}1 & 0 \\ 0 & 1 / 2\end{array}\right) \quad D\binom{x}{y}=\left(\begin{array}{l}1 \\ x \\ 1 / 2 y\end{array}\right)$

- scales the $x$-direction
- scales the $y$-direction by $Y_{2}$ [demo]

$E g: A=\frac{1}{6}\left(\begin{array}{ll}5 & 1 \\ 2 & 4\end{array}\right) \quad p(\lambda)=\lambda^{2}-\frac{3}{2} \lambda+\frac{1}{2}=(\lambda-1)\left(\lambda-\frac{1}{2}\right)$

$$
\lambda_{1}=1 \quad \omega_{1}=\binom{1}{1} \quad \lambda_{2}=\frac{1}{2} \quad \omega_{2}=\binom{-1}{2}
$$

Expand in the eigenbasis!

$$
A\left(x_{1} \omega_{1}+x_{2} \omega_{2}\right)=1 x_{1} \omega_{1}+\frac{1}{2} x_{2} \omega_{2}
$$

- scales the $w_{1}-d i r e c t i o n ~$ by 1
- Scales the $\omega_{2}$-direction by $Y_{2}$

Matrix From: $A=C D C^{-1} \quad C=\left(\begin{array}{cc}1 & -1 \\ 1 & 2\end{array}\right) \quad D=\left(\begin{array}{ll}1 & 0 \\ 0 & 1 / 2\end{array}\right)$

$\frac{(1) \text { multiply }}{\text { by } C^{-1}}$
[demo]

$\underbrace{\text { (3) }}_{\text {multiply }}$
by $C$

Eg: $A=\frac{1}{580}\left(\begin{array}{lll}503 & 73 & 269 \\ 207 & 1137 & -49 \\ 270 & -30 & 680\end{array}\right)$ has eigenbasis

$$
\omega_{1}=\left(\begin{array}{c}
-7 \\
2 \\
5
\end{array}\right) \quad \omega_{2}=\left(\begin{array}{c}
-1 \\
-9 \\
0
\end{array}\right) \quad \omega_{3}=\left(\begin{array}{c}
2 \\
-1 \\
3
\end{array}\right)
$$

and eigenvalues

$$
\begin{array}{lll}
\lambda_{1}=1 / 2 & \lambda_{2}=2 & \lambda_{3}=3 / 2
\end{array}
$$

Expand in the eigenbasis!

$$
A\left(x_{1} \omega_{1}+x_{2} \omega_{2}+x_{3} \omega_{3}\right)=\frac{1}{2} x_{1} \omega_{1}+2 x_{2} \omega_{2}+\frac{3}{2} x_{3} \omega_{3}
$$

- scales the $\omega_{1}$-direction by $\frac{1}{2}$
- scales the $\omega_{2}$-direction by 2 [demo]
- scales the $w_{3}-$ direction by $\frac{3}{2}$

