1. Decide if each statement is true or false, and explain why.
   a) A square matrix has no free variables.
   b) An invertible matrix has no free variables.
   c) An $m \times n$ matrix has at most $m$ pivots.
   d) A wide matrix (more columns than rows) must have a free variable.
   e) If $A$ is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

2. Consider the vectors
   \[ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]
   Draw the 16 linear combinations $cv + dw$ ($c, d = -1, 0, 1, 2$) as points in the $xy$-plane.

3. Certain vectors $v, w$ in $\mathbb{R}^2$ are drawn below. Express each of $b_1, b_2, b_3, b_4, b_5$ as a linear combination of $v, w$. Do not try to guess the coordinates of $v$ and $w$!

4. Consider the vectors
   \[ u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
   Draw a picture of all of the linear combinations $au + bv$ for real numbers $a, b$ satisfying $0 \leq a \leq 1$ and $0 \leq b \leq 1$. (This will be a shaded region in the $xy$-plane.)
5. Draw a picture of all vectors \( b \in \mathbb{R}^2 \) for which the equation
\[
\begin{pmatrix}
1 & 2 \\
-2 & -4 \\
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b
\]
is consistent. [Hint: the answer is a span!]

6. For each matrix \( A \) and vector \( b \), and express the solution set in the form
\[
p + \text{Span}\{???, ???\}
\]
for some vector \( p \). For instance,
\[
\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{---->} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \text{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}.
\]
[Hint: You found the parametric vector form in HW3#13.]

   a) \( A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

   b) \( A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix} \)

   c) \( A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix} \)

   d) \( A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \)

7. For each matrix \( A \) in Problem 6, write the solution set of \( Ax = 0 \) as a span. Does there exist a nontrivial solution? Do not do Gauss–Jordan elimination again!

8. When is the following system consistent?
\[
\begin{align*}
2x_1 + 2x_2 - x_3 &= b_1 \\
-4x_1 - 5x_2 + 5x_3 &= b_2 \\
6x_1 + x_2 + 12x_3 &= b_3
\end{align*}
\]
Your answer should be a single linear equation in \( b_1, b_2, b_3 \). [Hint: perform Gaussian elimination.]

   Explain the relationship between this equation
\[
\text{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 12 \end{pmatrix}\right\}.
\]
9. Let $A$ be a $3 \times 4$ matrix whose columns span the plane $x + y + z = 0$.
   a) Find a vector $b \in \mathbb{R}^3$ making the system $Ax = b$ consistent.
   b) Find a vector $b \in \mathbb{R}^3$ making the system $Ax = b$ inconsistent.

10. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

11. Give geometric descriptions of the following spans (line, plane, ...).
   a) Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$
   b) Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ -6 \end{pmatrix} \right\}$
   c) Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ -2 \\ -6 \end{pmatrix} \right\}$
   d) Span $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \\ -5 \\ 1 \\ 12 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix} \right\}$
   e) Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

   [Hint: for d), compare Problem 8.]

12. a) List five nonzero vectors contained in Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\}$.
   b) Is $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ contained in Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\}$?

   If so, express $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$.  

   c) Show that $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ is contained in Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\}$.
   d) Describe Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ geometrically.
   e) Find a vector not contained in Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\}$.  

13. Decide if each statement is true or false, and explain why.
   a) A vector \( b \) is a linear combination of the columns of \( A \) if and only if \( Ax = b \) has a solution.
   b) There is a matrix \( A \) such that \( Ax = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) has infinitely many solutions and \( Ax = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \) has exactly one solution.
   c) The zero vector is contained in every span.
   d) The matrix equation \( Ax = 0 \) can be consistent or inconsistent, depending on what \( A \) is.
   e) If the zero vector is a solution of a system of equations, then the system is homogeneous.
   f) If \( Ax = b \) has a unique solution, then \( A \) has a pivot in every column.
   g) If \( Ax = b \) is consistent, then the solution set of \( Ax = b \) is obtained by translating the solution set of \( Ax = 0 \).
   h) It is possible for \( Ax = b \) to have exactly 13 solutions.

14. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

   a) \( \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \)  
   b) \( \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \)  
   c) \( \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \)  
   d) \( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \)

[Hint: Compare Problem 7.]
15. Draw pictures of the null space and the column space of the following matrices. Be precise!

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]:

\[ A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \]:

\[ A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]:

16. Give examples of subsets \( V \) of \( \mathbb{R}^2 \) such that:

a) \( V \) is closed under addition and contains 0, but is not closed under scalar multiplication.

b) \( V \) is closed under scalar multiplication and contains 0, but is not closed under addition.

c) \( V \) is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.
17. Which of the following subsets of $\mathbb{R}^3$ are subspaces? If it is not a subspace, why not? If it is, write it as the column space or null space of some matrix.

a) The plane $\{(x, y, x) : x, y \in \mathbb{R}\}$.

b) The plane $\{(x, y, 1) : x, y \in \mathbb{R}\}$.

c) The set consisting of all vectors $(x, y, z)$ such that $xy = 0$.

d) The set consisting of all vectors $(x, y, z)$ such that $x \leq y$.

e) The span of $(1, 2, 3)$ and $(2, 1, -3)$.

f) The solution set of the system of equations
\[
\begin{align*}
  x + y + z &= 0 \\
  x - 2y - z &= 0.
\end{align*}
\]

g) The solution set of the system of equations
\[
\begin{align*}
  x + y + z &= 0 \\
  x - 2y - z &= 1.
\end{align*}
\]

18. Find a nonzero $2 \times 2$ matrix such that $A^2 = 0$.

19. a) Explain why $\text{Col}(AB)$ is contained in $\text{Col}(A)$.

b) Give an example where $\text{Col}(AB) \neq \text{Col}(A)$.

[Hint: use Problem 18.]

20. a) Explain why $\text{Nul}(AB)$ contains $\text{Nul}(B)$.

b) Give an example where $\text{Nul}(AB) \neq \text{Nul}(B)$.

[Hint: use Problem 18.]

21. Decide if each statement is true or false, and explain why.

a) The column space of an $m \times n$ matrix with $m$ pivots is a subspace of $\mathbb{R}^m$.

b) The null space of an $m \times n$ matrix with $n$ pivots is equal to $\mathbb{R}^n$.

c) If $\text{Col}(A) = \{0\}$, then $A$ is the zero matrix.

d) The column space of $2A$ equals the column space of $A$.

e) The null space of $A + B$ contains the null space of $A$.

f) If $U$ is an echelon form of $A$, then $\text{Nul}(U) = \text{Nul}(A)$.

g) If $U$ is an echelon form of $A$, then $\text{Col}(U) = \text{Col}(A)$.