Math 218D-1: Homework #14
due Wednesday, April 26, at 11:59pm

1. For each matrix $A$ of HW13#6:

   a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$
   b) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$
   c) $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$
   d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$
   e) $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

   find the singular value decomposition in the matrix form
   \[ A = U\Sigma V^T. \]

2. For each matrix $A$ of Problem 1, write down orthonormal bases for all four fundamental subspaces. (This can be read off from your answers to Problem 1.)

3. a) Let $A$ be an invertible $n \times n$ matrix. Show that the product of the singular values of $A$ equals the absolute value of the product of the (real and complex) eigenvalues of $A$ (counted with algebraic multiplicity).
   [Hint: Both equal $|\det(A)|$. What is $\det(A^T A)$?]

   b) Find an example of a $2 \times 2$ matrix $A$ with distinct positive eigenvalues that are not equal to any of the singular values of $A$.
   [Hint: One of the matrices in HW13#6 works.]

4. Let $S$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S = QDQ^T$ be an orthogonal diagonalization of $S$, where $D$ has diagonal entries $\lambda_1, \ldots, \lambda_n$.
   Show that $S = QDQ^T$ is a singular value decomposition if and only if $S$ is positive-semidefinite. [See HW13#9.]

5. Let $A$ be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

   a) Show that $A^{-1}$ has the same singular vectors as $A^T$, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$. [Hint: What is $A^+$?]

   b) Let $\lambda$ be an eigenvalue of $A$. Use HW13#12(b) and a) to show that $\sigma_n \leq |\lambda|$. It follows that the absolute values of all eigenvalues of $A$ are contained in the interval $[\sigma_n, \sigma_1]$. Compare Problem 3.
6. A certain $2 \times 2$ matrix $A$ has singular values $\sigma_1 = 2$ and $\sigma_2 = 1.5$. The right-singular vectors $v_1, v_2$ and the left-singular vectors $u_1, u_2$ are shown in the pictures below.
   a) Draw $Ax$ and $Ay$ in the picture on the right.
   b) Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit circle by $A$) in the picture on the right.

7. Consider the following $3 \times 2$ matrix $A$ and its SVD:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T.$$

Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit sphere by $A$) in the picture on the right.
8. Compute the pseudoinverse of each matrix of Problem 1.

9. a) Find a left inverse of the matrix

\[
A = \begin{pmatrix}
-3 & 11 \\
10 & -2 \\
1 & 5 \\
-4 & 6
\end{pmatrix}
\]

from Problem 1(c). (This is a matrix \( B \) such that \( BA \) is the identity.)

b) Find a right inverse of the matrix

\[
A = \begin{pmatrix}
9 & 7 & 10 & 8 \\
-13 & 1 & 5 & -6
\end{pmatrix}
\]

from Problem 1(d). (This is a matrix \( B \) such that \( AB \) is the identity.)

c) Explain why the matrix

\[
A = \begin{pmatrix}
1 & 3 \\
2 & 6
\end{pmatrix}
\]

from Problem 1(b) does not admit a left inverse or a right inverse.

10. Consider the matrix

\[
A = \begin{pmatrix}
3 & 7 & 1 & 5 \\
3 & 1 & 7 & 5 \\
6 & 2 & 2 & -2
\end{pmatrix}
\]

of Problem 8(e). Find the matrix \( P_V \) for projection onto \( V = \text{Row}(A) \) in two ways:

a) Multiply out \( P_V = A^+ A \).

b) In Problem 2 you found \( \text{Nul}(A) = \text{Span}\{v\} \) for \( v = (1, -1, -1, 1) \). Compute \( P_{V^\perp} = vv^T / v \cdot v \) and \( P_V = I_4 - P_{V^\perp} \).

Your answers to a) and b) should be the same, of course!

11. Let \( A \) be a matrix and let \( A^+ \) be its pseudoinverse. Match the subspaces on the left to the subspaces on the right:

\[
\begin{align*}
\text{Col}(A) & \quad \text{Col}(A^+) \\
\text{Nul}(A) & \quad \text{Nul}(A^+) \\
\text{Row}(A) & \quad \text{Row}(A^+) \\
\text{Nul}(A^T) & \quad \text{Nul}((A^+)^T)
\end{align*}
\]

What is the rank of \( A^+ \)?

12. What is the pseudoinverse of the \( m \times n \) zero matrix?
13. Consider the matrix \( A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \) of Problem 8(b).

a) Find all least-squares solutions of \( Ax = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) in parametric vector form.

b) Find the shortest least-squares solution \( \hat{x} = A^+ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \).

c) Draw your answers to a) and b) on the grid below.

14. Consider the following matrix holding 5 samples of 2 measurements each:

\[
A_0 = \begin{pmatrix} 22 & -12 & 24 & -29 & 20 \\ 1 & -11 & 37 & -17 & -35 \end{pmatrix}
\]

a) Subtract the means of the rows of \( A_0 \) to obtain the centered matrix \( A \).

b) Compute the covariance matrix \( S = \frac{1}{5-1} AA^T \). What is the total variance? What is the covariance of the first row with the second?

c) Compute the variance \( s(u)^2 \) of your data points in the directions

\[
u = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\]

d) Find the eigenvalues \( \lambda_1, \lambda_2 \) and unit eigenvectors \( u_1, u_2 \) of \( S \). What direction is the first principal component? What is the variance of \( A \) in that direction? (It should be larger than the variances you computed in c).)

e) Find the orthogonal projections of the columns of \( A \) onto the first principal component by computing the first summand \( \sigma_1 u_1 v_1^T \) of the SVD of \( A \). (Don’t forget to rescale by \( \sqrt{5-1} \)).

f) Draw the columns of \( A \), the first principal component you found in d), and the orthogonal projections you found in e) on a grid.
15. Let $A$ be a matrix with singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T.$$ 

Show that $A$ is a centered data matrix (rows sum to zero) if and only if the entries of each right singular vector $v_i$ sum to zero. 

[Hint: Multiply by the ones vector $1 = (1, 1, \ldots, 1)$.]

16. Let $A$ be a matrix with singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$ 

Recall that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is $\sigma_1$, and is achieved at $x = v_1$.

a) Show that the maximum value of $\|Ax\|$ subject to the conditions $\|x\| = 1$ and $x \cdot v_1 = 0$ is equal to $\sigma_2$, and is achieved at $x = v_2$.

[Hint: If $x \cdot v_1 = 0$ then $Ax = A'x$ for $A' = \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T + \cdots + \sigma_r u_r v_r^T$.]

b) More generally, show that the maximum value of $\|Ax\|$ subject to the conditions $\|x\| = 1$ and $x \cdot v_1 = 0$, $x \cdot v_2 = 0$, \ldots, $x \cdot v_j = 0$ is equal to $\sigma_{j+1}$, and is achieved at $x = v_{j+1}$.

c) If $A$ has full column rank, show that the minimum value of $\|Ax\|$ subject to $\|x\| = 1$ is equal to $\sigma_r$, and is achieved at $x = v_r$.

In the language of principal component analysis, this says that $v_2$ is the direction of second-largest variance, etc.

17. Decide if each statement is true or false, and explain why.

a) If $A$ is a matrix of rank $r$, then $A$ is a linear combination of $r$ rank-1 matrices.

b) If $A$ is a matrix of rank 1, then $A^+$ is a scalar multiple of $A^T$.

c) If $A = U\Sigma V^T$ is the SVD of $A$, then the SVD of $A^+$ is $A^+ = V \Sigma^+ U^T$. 