Math 218D-1: Homework #10

due Wednesday, March 29, at 11:59pm

1. For each $2 \times 2$ matrix $A$, i) compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$. Use this to ii) find all real eigenvalues, and iii) find a basis for each eigenspace, using HW9#13 when applicable. iv) Draw and label each eigenspace. v) Is the matrix diagonalizable (over the real numbers)?

   a) \[
   \begin{pmatrix}
   1 & -2 \\
   1 & 4
   \end{pmatrix}
   \]
   b) \[
   \begin{pmatrix}
   -1 & 1 \\
   -9 & 5
   \end{pmatrix}
   \]
   c) \[
   \begin{pmatrix}
   3 & 0 \\
   0 & 3
   \end{pmatrix}
   \]
   d) \[
   \begin{pmatrix}
   1 & 1 \\
   -1 & 1
   \end{pmatrix}
   \]
   e) \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 0
   \end{pmatrix}
   \]

2. For each matrix $A$, i) find all real eigenvalues of $A$, and ii) find a basis for each eigenspace. iii) Is the matrix diagonalizable (over the real numbers)?

   You will probably want to use a computer algebra system to find the roots of the characteristic polynomial. To do so in Sympy, you would type something like:

   ```python
   print(roots(-x**3 + 13/4*x + 3/2, multiple=True))
   ```

   # [-1.5, -0.5, 2.0]

   a) \[
   \begin{pmatrix}
   -1 & 7 & 5 \\
   0 & 1 & -2 \\
   0 & 1 & 4
   \end{pmatrix}
   \]
   b) \[
   \begin{pmatrix}
   7 & 12 & 12 \\
   -8 & -13 & -12 \\
   4 & 6 & 5
   \end{pmatrix}
   \]
   c) \[
   \begin{pmatrix}
   -14 & -7 & -12 \\
   6 & 2 & 3
   \end{pmatrix}
   \]

   Optional (if you want more practice):

   d) \[
   \begin{pmatrix}
   -11 & -54 & 10 \\
   -2 & -7 & 2 \\
   -21 & -90 & 20
   \end{pmatrix}
   \]
   e) \[
   \begin{pmatrix}
   2 & 0 & 0 \\
   0 & 2 & 0 \\
   0 & 0 & 2
   \end{pmatrix}
   \]
   
   f) \[
   \begin{pmatrix}
   13 & 18 & -18 \\
   -12 & -17 & 18 \\
   -4 & -6 & 7
   \end{pmatrix}
   \]
   g) \[
   \begin{pmatrix}
   -10 & 28 & -18 & -76 \\
   -1 & 9 & -6 & -2 \\
   4 & -8 & 7 & 26 \\
   0 & 2 & -2 & 4
   \end{pmatrix}
   \]

3. Let $V$ be the plane $x + y + z = 0$, and let $R_V = I_3 - 2P_{V\perp}$ be the reflection matrix over $V$, as in HW9#5. Find an eigenbasis for $R_V$ without doing any computations. Is $R_V$ diagonalizable?
4. The Fibonacci numbers are defined recursively as follows:

\[ F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0). \]

The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, \ldots In this problem, you will find a closed formula (as opposed to a recursive formula) for the \( n \)th Fibonacci number by solving a difference equation.

a) Let \( v_n = \left( \begin{array}{c} F_{n+1} \\ F_n \end{array} \right) \), so \( v_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \), \( v_1 = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \), etc. Find a state change matrix \( A \) such that \( v_{n+1} = Av_n \) for all \( n \geq 0 \).

b) Show that the eigenvalues of \( A \) are \( \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \) and \( \lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \), with corresponding eigenvectors \( w_1 = \left( \begin{array}{c} -1 \\ \lambda_2 \end{array} \right) \) and \( w_2 = \left( \begin{array}{c} -1 \\ \lambda_1 \end{array} \right) \).

[Hint: Check that \( Aw_i = \lambda_i w_i \) using the relations \( \lambda_1 \lambda_2 = -1 \) and \( \lambda_1 + \lambda_2 = 1 \).]

c) Expand \( v_0 \) in this eigenbasis: that is, find \( x_1, x_2 \) such that \( v_0 = x_1 w_1 + x_2 w_2 \). (It helps to write \( x_1, x_2 \) in terms of \( \lambda_1, \lambda_2 \).)

d) Multiply \( v_0 = x_1 w_1 + x_2 w_2 \) by \( A^n \) to show that

\[ F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \]

e) Use this formula to explain why \( F_{n+1}/F_n \) approaches the golden ratio when \( n \) is large.
5. Pretend that there are three Red Box kiosks in Durham. Let \( x_t, y_t, z_t \) be the number of copies of Prognosis Negative at each of the three kiosks, respectively, on day \( t \). Suppose in addition that a customer renting a movie from kiosk \( i \) will return the movie the next day to kiosk \( j \), with the following probabilities:

<table>
<thead>
<tr>
<th>Returning to kiosk</th>
<th>Renting from kiosk</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 30% 40% 50%</td>
</tr>
<tr>
<td>2</td>
<td>2 30% 40% 30%</td>
</tr>
<tr>
<td>3</td>
<td>3 40% 20% 20%</td>
</tr>
</tbody>
</table>

For instance, a customer renting from kiosk 3 has a 50\% probability of returning it to kiosk 1.

a) Let \( v_t = (x_t, y_t, z_t) \). Find the state change matrix \( A \) such that \( v_{t+1} = Av_t \).

b) Diagonalize \( A \). What are its eigenvalues?

[Hint: \( A \) is a stochastic matrix, so you know one eigenvalue by HW9#14(c).]

c) If you start with a total of 1 000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk? Does it matter what the initial state is?

This is an example of a stochastic process, and is an important application of eigenvalues and eigenvectors.

6. For each 2\( \times \)2 matrix \( A \) in Problem 1, if \( A \) is diagonalizable, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

7. For each matrix \( A \) in Problem 2, if \( A \) is diagonalizable, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

8. Consider the matrix

\[
A = \begin{pmatrix}
4 & -3 & 0 \\
2 & -1 & 0 \\
1 & -1 & 1
\end{pmatrix}.
\]

a) Find a diagonal matrix \( D \) and an invertible matrix \( C \) such that \( A = CDC^{-1} \).

b) Find a different diagonal matrix \( D' \) and a different invertible matrix \( C' \) such that \( A = C'D'C'^{-1} \).

[Hint: Try re-ordering the eigenvalues.]
9. Compute the matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors
\[
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.
\]
(There is only one such matrix.)

10. Let \(A\) and \(B\) be \(n \times n\) matrices, and let \(v_1, \ldots, v_n\) be a basis of \(\mathbb{R}^n\).
   a) Suppose that each \(v_i\) is an eigenvector of both \(A\) and \(B\). Show that \(AB = BA\).
   b) Suppose that each \(v_i\) is an eigenvector of both \(A\) and \(B\) with the same eigenvalue. Show that \(A = B\).
   [Hint: use the matrix form of diagonalization.]

11. Let \(A\) be an \(n \times n\) matrix, and let \(C\) be an invertible \(n \times n\) matrix. Prove that the characteristic polynomial of \(CAC^{-1}\) equals the characteristic polynomial of \(A\).
   In particular, \(A\) and \(CAC^{-1}\) have the same eigenvalues, the same determinant, and the same trace. They are called similar matrices.

12. Let \(A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\). Find a closed formula for \(A^n\): that is, an expression of the form
\[
A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},
\]
where \(a_{ij}(n)\) is a function of \(n\).

13. A certain \(2 \times 2\) matrix \(A\) has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.
   a) Draw \(Av\), \(A^2v\), and \(Aw\).
   b) Compute the limit of \(A^n v/\|A^n v\|\) as \(n \to \infty\).
14. A certain diagonalizable $2 \times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $w_1, w_2$ pictured below, and $D = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

a) Draw $C^{-1}v$ on the left.

b) Draw $DC^{-1}v$ on the left.

c) Draw $Av = CDC^{-1}v$ on the right.

d) What happens to $A^nv$ as $n \to \infty$?