MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

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Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a simple calculator for doing arithmetic, but you should not need one. You may bring a 3 × 5-inch note card covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [15 points]

Consider the matrix

\[
A = \begin{pmatrix}
3 & 2 & 1 \\
-3 & -3 & -2 \\
9 & 7 & 2
\end{pmatrix}.
\]

a) Find a lower-unitriangular matrix \( L \) and a matrix \( U \) in REF such that \( A = LU \).

\[
L = \begin{pmatrix}
\end{pmatrix} \quad U = \begin{pmatrix}
\end{pmatrix}
\]

b) Use the LU decomposition you found in a) to solve the equation \( Ax = (0, 2, 0) \).

\[
x = \begin{pmatrix}
\end{pmatrix}
\]

c) Briefly explain why one would want to use an LU decomposition to solve \( Ax = b \).

d) Express \( L \) as a product of three elementary matrices.

\[
L = \begin{pmatrix}
\end{pmatrix} \begin{pmatrix}
\end{pmatrix} \begin{pmatrix}
\end{pmatrix}
\]
[Scratch work for Problem 1]
Problem 2. [25 points]

Consider the system of equations
\[
\begin{align*}
x_1 - 2x_2 - x_4 &= 2 \\
-x_1 + 2x_2 + x_3 + 2x_4 &= -3 \\
3x_1 - 6x_2 - x_3 - 4x_4 &= 7.
\end{align*}
\]

a) Express this system of equations as a matrix equation \( Ax = b \):
\[
\begin{pmatrix}
& & & & 1 & -2 & -1 & 0 & 0 \\
1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
3 & -6 & 0 & 0 & 0 & -1 & -2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-3 \\
7
\end{pmatrix}
\]

b) Perform Gauss–Jordan elimination on the augmented matrix \( A | b \) to produce a matrix in reduced row echelon form. Please write all row operations that you perform.
\[
\begin{pmatrix}
& & & & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
3 & -6 & 0 & 0 & 0 & -1 & -2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-3 \\
7
\end{pmatrix}
\]

c) The free variables(s) are \[ \boxed{} \] and the rank of \( A \) is \[ \boxed{} \].
Scratch work for Problem 2
(Problem 2, continued)

d) Write the solution set of this system of equations in parametric vector form.

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix} = \begin{pmatrix}
\end{pmatrix} + \begin{pmatrix}
\end{pmatrix}
\]

e) The solution set is a (circle one) \text{ line} \quad \text{ plane} \quad \text{ space} \quad \text{ in (fill in the blank)} \ R^\text{ }.

f) Find a basis for \text{ Nul}(A).

\[
\begin{pmatrix}
\end{pmatrix}
\]

g) Write down any \textit{nontrival} solution of \text{Ax} = 0.

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix} = \begin{pmatrix}
\end{pmatrix}
\]
[Scratch work for Problem 2]
Problem 3. 

Consider the vectors 

\[ v_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}. \]

a) Find a linear relation among \( v_1, v_2, v_3, v_4 \).

\[ 0 = \]

b) The set \( \{v_1, v_2, v_3, v_4\} \) is (circle one) \( \text{linearly dependent} \) \( \text{linearly independent} \).

c) Which one of the following two vectors is in \( \text{Span}\{v_1, v_2, v_3, v_4\} \)?
(Circle the correct one.)

\[ u = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \]

d) Express \( (2, -3, 7) \) as a linear combination of \( v_1, v_2, v_3, v_4 \) in two different ways.

\[ \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} \boxed{1} \\ \boxed{-1} \\ \boxed{3} \end{pmatrix} + \begin{pmatrix} \boxed{-2} \\ \boxed{2} \\ \boxed{-6} \end{pmatrix} + \begin{pmatrix} \boxed{0} \\ \boxed{1} \\ \boxed{-1} \end{pmatrix} + \begin{pmatrix} \boxed{-1} \\ \boxed{2} \\ \boxed{-4} \end{pmatrix} \]

\[ \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} \boxed{1} \\ \boxed{-1} \\ \boxed{3} \end{pmatrix} + \begin{pmatrix} \boxed{-2} \\ \boxed{2} \\ \boxed{-6} \end{pmatrix} + \begin{pmatrix} \boxed{0} \\ \boxed{1} \\ \boxed{-1} \end{pmatrix} + \begin{pmatrix} \boxed{-1} \\ \boxed{2} \\ \boxed{-4} \end{pmatrix} \]

e) Which of the following sets form a basis for \( \text{Span}\{v_1, v_2, v_3, v_4\} \)?
(Circle all that apply.)

\[ \{v_1, v_2\} \quad \{v_1, v_2, v_3\} \quad \{v_2, v_3\} \quad \{v_2, v_4\} \quad \{v_3, v_4\} \quad \{v_2, v_3, v_4\} \]

f) \( \text{dim}(\text{Span}\{v_1, v_2, v_3\}) = \boxed{ } \)
[Scratch work for Problem 3]
Problem 4.  

For a certain $2 \times 2$ matrix $A$ and vector $b \in \mathbb{R}^2$, the solution set of $Ax = b$ is drawn in the picture on the left, and $b$ is drawn in the picture on the right.

a) Draw $\text{Nul}(A)$ in the picture on the left.

b) Draw $\text{Col}(A)$ in the picture on the right.

c) Draw a vector $b'$ in the picture on the right that makes the system $Ax = b'$ inconsistent.
[Scratch work for Problem 4]
Problem 5.

For each of the following subsets,

- determine if the subset is a subspace.

If it is not a subspace,

- explain why not.

If it is a subspace,

- express it as the null space or the column space of a matrix, and
- find a basis.

\[ V_1 = \left\{ \begin{pmatrix} 2(a+c) - b \\ a - 2b + c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \]

\[ V_2 = \left\{ \begin{pmatrix} a + b \\ 2 \\ 2a - 2b \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \]

\[ V_3 = \text{the } x\text{-axis in the } xy\text{-plane} \]

\[ V_4 = \text{all vectors } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } x \geq y \]
[Scratch work for Problem 5]
Problem 6. [15 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead. You need not justify your answers.

a) A $2 \times 2$ matrix whose column space is a point.

b) A $2 \times 2$ matrix whose null space is all of $\mathbb{R}^2$.

c) An invertible $3 \times 3$ matrix $A$ such that the solution set of $Ax = 0$ is the $x$-axis.

d) A $2 \times 2$ matrix whose columns are linearly dependent.

e) A $2 \times 2$ matrix $A$ such that the solution set of $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the point $\{-1\}$, and such that the system $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is inconsistent.
[Scratch work for Problem 6]