Math 218D Problem Session: Week 8

October 21, 2022

1. Gram-Schmidt and QR

The purpose of the Gram–Schmidt process is to replace a basis $\{v_1, \ldots, v_k\}$ of a subspace V of $\mathbf{R}^n$ with an **orthogonal basis** of V (a basis whose vectors are an orthogonal set).

The vectors $v_1 = (1,2,-2)$, $v_2 = (1,1,1)$ form a basis for a plane V in $\mathbf{R}^3$. Set

$$u_1 = v_1$$
$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.$$

These two vectors are the output of the Gram–Schmidt process.

a) Compute $\frac{u_1}{\|u_1\|}$ and $\frac{u_2}{\|u_2\|}$, and confirm that $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}\right\}$ is an orthonormal set of vectors (you need to compute 3 dot products).

b) We can find the QR decomposition of $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$ by setting

$$Q = \begin{pmatrix} | & | \\ \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \\ | & | \end{pmatrix}.$$

Then $A = QR$ for some upper-triangular matrix R, and you saw a formula for R in lecture. Here is another way to find R:

$$R = Q^T A.$$

Use this to compute R, and confirm that $A = QR$ by multiplying Q times R.

Note: The method of finding R given in lecture is much faster, as it involves only book-keeping your work from finding Q.

c) Explain why this formula for R worked, i.e. why $A = QR$ had to imply that $Q^T A = R$.

d) Explain how you could compute the projection matrix P_V using Q. (You do not need to do the computation.)

e) Find the least-squares solution of $Ax = (1,1,0)$ using $R\widehat{x} = Q^T b$.

2. Another Gram–Schmidt

a) Apply the Gram–Schmidt process to the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

to obtain an orthogonal set $\{u_1, u_2, u_3\}$.

b) Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

c) Consider the vector $b = (1,1,1)$. Since $\{u_1, u_2, u_3\}$ is a basis for $\mathbf{R}^3$, there are scalars x_1, x_2, x_3 such that $b = x_1u_1 + x_2u_2 + x_3u_3$. Solve for these scalars by taking the dot product of this equation with each of u_1, u_2, u_3, giving 3 equations

$$b \cdot u_i = (x_1u_1 + x_2u_2 + x_3u_3) \cdot u_i \quad \text{for} \quad i = 1,2,3.$$

(These equations simplify dramatically when you compute the dot products.)

d) Explain how you could instead solve for these scalars using the formula $QQ^T = P_{\mathbf{R}^3} = I_3$.
Hint: Note that $b = Q(Q^T b)$.

3. Some quick determinants

Compute the determinants of the following matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 10 & 17 \\ 0 & 2 & \pi \\ 0 & 0 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ **h)** $\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}^{20}$

4. Some determinants with variables

a) Compute the determinant of each of $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$, A^2, A^{-1}, and $A - xI_2$. Find the two values of x so that $\det(A - xI_2) = 0$.

b) Compute the determinant of

$$\begin{pmatrix} 1-x & 1 & 1 \\ 2 & 2-x & 2 \\ 1 & 2 & 3-x \end{pmatrix}.$$

This is a polynomial in the variable x—what degree is the polynomial?

5. Signs of determinants

We gave a geometric interpretation of the absolute value of a determinant in lecture. In this problem we will investigate what the *sign* of a determinant means geometrically. (The *sign* of a number is +1 if the number is positive and −1 if it is negative.)

a) Draw the vectors $u = (1, -1)$, $v = (2, 3)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$?

b) Draw the vectors $u = (-1, 2)$, $v = (1, 1)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$?

Let u, v, w be vectors in $\mathbf{R}^3$. With your right hand, point your index finger in the direction of u, your middle finger in the direction of v, and your thumb in the direction of w. We say that u, v, w are in *right-hand order* if, when you point your thumb at your face, your middle finger is counterclockwise of your index finger. Otherwise, the vectors are in *left-hand order*.

c) Are the vectors $u = (0, 1, 0)$, $v = (1, 1, 0)$, $w = (1, 1, 1)$ in right-hand order or left-hand order?

d) Are the vectors $u = (1, 1, 0)$, $v = (0, 1, 0)$, $w = (1, 1, 1)$ in right-hand order or left-hand order?

e) What is the sign of the determinants of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}?$$

f) What do you think the sign of the 3×3 determinant has to do with right-hand order?