1. **Gram-Schmidt and QR**

The purpose of the Gram–Schmidt process is to replace a basis \( \{v_1, \ldots, v_k\} \) of a subspace \( V \) of \( \mathbb{R}^n \) with an **orthogonal basis** of \( V \) (a basis whose vectors are an orthogonal set).

The vectors \( v_1 = (1, 2, -2) \), \( v_2 = (1, 1, 1) \) form a basis for a plane \( V \) in \( \mathbb{R}^3 \). Set

\[
\begin{align*}
  u_1 &= v_1 \\
  u_2 &= v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.
\end{align*}
\]

These two vectors are the output of the Gram–Schmidt process.

**a)** Compute \( \frac{u_1}{\|u_1\|} \) and \( \frac{u_2}{\|u_2\|} \), and confirm that \( \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|} \right\} \) is an orthonormal set of vectors (you need to compute 3 dot products).

**b)** We can find the QR decomposition of \( A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \) by setting

\[
Q = \begin{pmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \\ \|u_1\| & \|u_2\| \end{pmatrix}.
\]

Then \( A = QR \) for some upper-triangular matrix \( R \), and you saw a formula for \( R \) in lecture. Here is another way to find \( R \):

\[
R = Q^T A.
\]

Use this to compute \( R \), and confirm that \( A = QR \) by multiplying \( Q \) times \( R \).

**Note:** The method of finding \( R \) given in lecture is much faster, as it involves only book-keeping your work from finding \( Q \).

**c)** Explain why this formula for \( R \) worked, i.e. why \( A = QR \) had to imply that \( Q^T A = R \).

**d)** Explain how you could compute the projection matrix \( P_V \) using \( Q \). (You do not need to do the computation.)

**e)** Find the least-squares solution of \( Ax = (1, 1, 0) \) using \( R\vec{x} = Q^T b \).
2. Another Gram–Schmidt
   
a) Apply the Gram–Schmidt process to the vectors
   \[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \]
   to obtain an orthogonal set \{u_1, u_2, u_3\}.

b) Find the QR decomposition of \[ A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \]

c) Consider the vector \( b = (1, 1, 1) \). Since \{u_1, u_2, u_3\} is a basis for \( \mathbb{R}^3 \), there are scalars \( x_1, x_2, x_3 \) such that \( b = x_1u_1 + x_2u_2 + x_3u_3 \). Solve for these scalars by taking the dot product of this equation with each of \( u_1, u_2, u_3 \), giving 3 equations
   \[ b \cdot u_i = (x_1u_1 + x_2u_2 + x_3u_3) \cdot u_i \quad \text{for} \quad i = 1, 2, 3. \]
   (These equations simplify dramatically when you compute the dot products.)

d) Explain how you could instead solve for these scalars using the formula \( QQ^T = P_{\mathbb{R}^3} = I_3 \).
   \textbf{Hint:} Note that \( b = Q(Q^T b) \).
3. Some quick determinants

Compute the determinants of the following matrices:

\[ \begin{align*}
\text{a)} & \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} & \text{b)} & \quad \begin{pmatrix} 1 & 10 & 17 \\ 0 & 2 & \pi \\ 0 & 0 & 3 \end{pmatrix} & \text{c)} & \quad \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \\
\text{d)} & \quad \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} & \text{e)} & \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \text{f)} & \quad \begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \\
\text{g)} & \quad \begin{pmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \text{h)} & \quad \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}^{20} 
\end{align*} \]
4. Some determinants with variables
   
a) Compute the determinant of each of \( A = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \end{pmatrix}, A^2, A^{-1}, \) and \( A - xI_2 \). Find the two values of \( x \) so that \( \det(A - xI_2) = 0 \).
   
b) Compute the determinant of
   \[
   \begin{pmatrix}
   1-x & 1 & 1 \\
   2 & 2-x & 2 \\
   1 & 2 & 3-x \\
   \end{pmatrix}.
   \]
   This is a polynomial in the variable \( x \)—what degree is the polynomial?
5. Signs of determinants

We gave a geometric interpretation of the absolute value of a determinant in lecture. In this problem we will investigate what the sign of a determinant means geometrically. (The sign of a number is +1 if the number is positive and −1 if it is negative.)

a) Draw the vectors \( u = (1, -1), \ v = (2, 3) \). Is \( v \) clockwise or counterclockwise from \( u \)? What is the sign of the determinant of \( \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \)?

b) Draw the vectors \( u = (-1, 2), \ v = (1, 1) \). Is \( v \) clockwise or counterclockwise from \( u \)? What is the sign of the determinant of \( \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \)?

Let \( u, v, w \) be vectors in \( \mathbb{R}^3 \). With your right hand, point your index finger in the direction of \( u \), your middle finger in the direction of \( v \), and your thumb in the direction of \( w \). We say that \( u, v, w \) are in right-hand order if, when you point your thumb at your face, your middle finger is counterclockwise of your index finger. Otherwise, the vectors are in left-hand order.

c) Are the vectors \( u = (0, 1, 0), \ v = (1, 1, 0), \ w = (1, 1, 1) \) in right-hand order or left-hand order?

d) Are the vectors \( u = (1, 1, 0), \ v = (0, 1, 0), \ w = (1, 1, 1) \) in right-hand order or left-hand order?

e) What is the sign of the determinants of

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

f) What do you think the sign of the 3 × 3 determinant has to do with right-hand order?