Math 218D Problem Session: Week 7

October 14, 2022

1. Linear Regression

Let us find the line $y = Cx + D$ which best fits the data points $(1,3)$, $(2,2)$, $(-2,1)$ (in the least-squares sense). If these points were collinear, then the coefficients C and D would solve the equation

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

But there is no solution to this system, as these 3 data points are not collinear. Instead, we will find the *least-squares solution* $\widehat{x} = \binom{C}{D}$, i.e., the solution of

$$A^T A\widehat{x} = A^T b.$$

a) Compute $A^T A$ and $A^T b$, and solve for the least-squares solutions $\widehat{x} = \binom{C}{D}$.

b) Plot the data points and the least-squares line $y = Cx + D$.

c) Where do the numbers in the vector $b - A\widehat{x}$ appear in the picture?

d) Compute the *error* $\|b - A\widehat{x}\|$.

2. Least Squares Practice

Find all least-squares solutions $\widehat{x}$ of each of the following systems of equations $Ax = b$, and compute the projection b_V of b onto $V = \text{Col}(A)$ and the minimum value of $\|A\widehat{x} - b\|$.

a) $\begin{pmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 8 & 2 & 3 \\ -3 & 4 & 4 \\ -2 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} -11 \\ 4 \\ 2 \end{pmatrix}$

3. Orthogonal matrices

An *orthogonal matrix* is a *square* matrix Q such that $Q^T Q = I_n$. Which of the following matrices are orthogonal?

a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4. Rotation and reflection

A *rotation matrix* $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is an example of an orthogonal matrix.

a) Verify that R_θ is an orthogonal matrix by checking $R_\theta^T R_\theta = I_2$.

b) Draw the vectors $R_{\pi/6}\binom{1}{0}$ and $R_{\pi/6}\binom{1}{1}$.

c) Using dot products, compute the angle between the rotated vectors $R_{\pi/6}\binom{1}{0}$ and $R_{\pi/6}\binom{1}{1}$. Confirm that this is the same as the angle between the two vectors $\binom{1}{0}$ and $\binom{1}{1}$.

This is an example of a general phenomenon: multiplying by an orthogonal matrix preserves angles and lengths.

Consider a line $L = \text{Span}\{v\}$ in $\mathbf{R}^3$, and the orthogonal complement plane $V = L^\perp$. The *reflection matrix* for *reflection across* V is the orthogonal matrix

$$Q = I_3 - 2P_L,$$

where P_L is the projection matrix for L.

d) When $L = \text{Span}\{(0,0,1)\}$, compute the reflection matrix Q. Draw the line L and the plane V. Draw the vector $(1,-1,1)$, and compute and draw the projection $P_L(1,-1,1)$ and the reflection $Q(1,-1,1)$.

e) Confirm that *any* reflection matrix $Q = I_3 - 2P_L$ is an orthogonal matrix by showing that $Q^T Q = (I_3 - 2P_L)^T(I_3 - 2P_L)$ equals I_3.
[**Hint:** Remember that $P_L^2 = P_L$ and $P_L^T = P_L$.]