Math 218D Problem Session: Week 6

October 7, 2022

1. Projection onto a line

For each of the following pairs of vectors b and v,

(1) compute the orthogonal projection of b onto the line $V = \text{Span}\{v\}$,

(2) draw V and the three vectors $b, b_V, b_{V^\perp}$, and

(3) compute the projection matrix $P_V = vv^T/(v^T v)$.

a) $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **b)** $b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **c)** $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

2. Planes and normal vectors

The subspace $V = \text{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) Place the vectors $(1,1,2)$, $(1,3,1)$ into the rows of a 2×3 matrix A—this means that $\text{Row}(A) = V$. Find a basis for $\text{Nul}(A)$. Since

$$V^\perp = \text{Row}(A)^\perp = \text{Nul}(A),$$

you have found a basis vector $v = (a,b,c)$ for the line $V^\perp$.

In other words, you have found a basis for $V^\perp$ by solving the two orthogonality equations

$$(a,b,c)\cdot(1,1,2) = a+b+2c = 0,$$
$$(a,b,c)\cdot(1,3,1) = a+3b+c = 0.$$

b) Confirm that V is the plane $\{(x,y,z)\in\mathbf{R}^3 : ax+by+cz=0\}$, by showing that both $(1,1,2)$ and $(1,3,1)$ solve this equation. *The coefficients of a plane's equation make a normal vector for the plane.*

c) Find the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1,1,1)$ with respect to the plane V and the orthogonal line $V^\perp$.

Hint: It is easier to compute $b_{V^\perp}$, as it is the projection of b onto the line $V^\perp$ spanned by the vector $v = (a,b,c)$.

3. Orthogonal projections, under the hood

Consider the plane

$$V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}$$

in $\mathbf{R}^4$. We will find the orthogonal projection of $b = (1,-1,-3,-5)$ onto V, "by hand." This is the vector $b_V \in V$ satisfying $b_{V^\perp} = b - b_V \in V^\perp$.

Since b_V is in V, there exist scalars $\widehat{x}_1, \widehat{x}_2$ such that

$$b_V = \widehat{x}_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix}.$$

We will compute the orthogonal projection by solving for these scalars.

The vector $b_{V^\perp}$ is orthogonal to every vector in V. In particular, it is orthogonal to both $(1,1,1,1)$ and $(1,2,3,4)$. We get two equations:

$$\begin{aligned}(1,1,1,1)\cdot b_{V^\perp} &= 0,\\ (1,2,3,4)\cdot b_{V^\perp} &= 0.\end{aligned}$$

Expanding

$$b_{V^\perp} = b - b_V = \begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix} - \widehat{x}_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \widehat{x}_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix},$$

we can rewrite these two equations as

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\left(\widehat{x}_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\cdot\left(\widehat{x}_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right) = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix}.$$

a) By computing the dot products, convert this into two linear equations in the two unknowns $\widehat{x}_1$ and $\widehat{x}_2$.

b) Solve for $\widehat{x}_1$ and $\widehat{x}_2$, and compute the orthogonal projection

$$b_V = \widehat{x}_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix}.$$

c) Confirm that the vector $b_{V^\perp} = b - b_V$ is orthogonal to V by checking that

$$b_{V^\perp}\cdot(1,1,1,1) = 0 \quad \text{and} \quad b_{V^\perp}\cdot(1,2,3,4) = 0.$$

d) Write down a matrix A whose column are the two vectors which span V, and compute $A^T A$, the *matrix of column dot products*. Compute the vector $A^T b$. Explain where the matrix equation $A^T A\widehat{x} = A^T b$ (the *normal equation*) appears in **a)** and **b)**, and also where the product $b_V = A\widehat{x}$ appears.

4. Projection matrices for planes

Consider the plane

$$V = \operatorname{Span}\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right\}$$

in $\mathbf{R}^4$.

a) Compute the projection matrix P_V for the subspace V. (Feel free to use a computer.)

b) Explain why the first two columns of $I_4 - P_V$ form a basis for $V^\perp$.

c) Use your answer to **b)** to describe the plane V via *two* implicit equations:

$$V = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \mathbf{R}^4 \ : \ \begin{aligned} c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 &= 0 \\ d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 &= 0 \end{aligned} \right\}.$$

What are the coefficients (c_1, c_2, c_3, c_4) and (d_1, d_2, d_3, d_4), and why? Confirm that every vector in V satisfies these equations by checking that both $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ do.

5. Projection matrices for lines

For each line L, compute the projection matrix P_L.

$$\textbf{a) } L = \mathrm{Span}\{(1,1)\} \qquad \textbf{b) } L = \mathrm{Span}\{(1,2,3)\}$$

$$\textbf{c) } L = \big\{(x,y,z) \in \mathbf{R}^3 : 2x + y + z = 0\big\}^{\perp}$$

6. Some mistakes to avoid

Here is a false "fact":

Every projection matrix P_V equals the identity matrix I_n.

Here is a false "proof":

$$P_V = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = (AA^{-1})((A^T)^{-1}A^T) = I_n \cdot I_n = I_n.$$

a) What is wrong would this proof?

b) In what case would this proof be correct?

Now consider the subspace $V = \mathrm{Col}(A)$ for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) It would be *incorrect* to say that $P_V = A(A^TA)^{-1}A^T$ is the projection matrix onto V. Why?

Hint: Try computing P_V—what goes wrong?

d) Find a matrix B so that $P_V = B(B^TB)^{-1}B^T$ is the projection matrix onto V—you do not need to compute P_V.