1. **Projection onto a line**

For each of the following pairs of vectors \( b \) and \( v \),

1. compute the orthogonal projection of \( b \) onto the line \( V = \text{Span}\{v\} \),
2. draw \( V \) and the three vectors \( b, b_V, b_{V\perp} \), and
3. compute the projection matrix \( P_V = vv^T / (v^Tv) \).

**Solution.**

a) \[ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ b_V = \frac{b \cdot v}{v \cdot v} v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; b_{V\perp} = b - b_V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; P_V = \frac{vv^T}{v^Tv} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

b) \[ b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ b_V = \frac{b \cdot v}{v \cdot v} v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; b_{V\perp} = b - b_V = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; P_V = \frac{vv^T}{v^Tv} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

c) \[ b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \]

\[ b_V = \frac{b \cdot v}{v \cdot v} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; b_{V\perp} = b - b_V = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; P_V = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \]
2. Planes and normal vectors

The subspace \( V = \text{Span}\{(1, 1, 2), (1, 3, 1)\} \) of \( \mathbb{R}^3 \) is a plane.

a) Place the vectors \((1, 1, 2), (1, 3, 1)\) into the rows of a \( 2 \times 3 \) matrix \( A \)—this means that \( \text{Row}(A) = V \). Find a basis for \( \text{Nul}(A) \). Since \( V \perp = \text{Row}(A) \perp = \text{Nul}(A) \), you have found a basis vector \( v = (a, b, c) \) for the line \( V \perp \).

In other words, you have found a basis for \( V \perp \) by solving the two orthogonality equations

\[
(a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0,
(a, b, c) \cdot (1, 3, 1) = a + 3b + c = 0.
\]

b) Confirm that \( V \) is the plane \( \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\} \), by showing that both \((1, 1, 2)\) and \((1, 3, 1)\) solve this equation. The coefficients of a plane’s equation make a normal vector for the plane.

c) Find the orthogonal decomposition \( b = b_v + b_{V \perp} \) of the vector \( b = (1, 1, 1) \) with respect to the plane \( V \) and the orthogonal line \( V \perp \).

Hint: It is easier to compute \( b_{V \perp} \), as it is the projection of \( b \) onto the line \( V \perp \) spanned by the vector \( v = (a, b, c) \).

Solution.

a) We perform Gauss–Jordan elimination:

\[
A = \begin{pmatrix}
1 & 1 & 2 \\
1 & 3 & 1 \\
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
1 & 0 & 5/2 \\
0 & 1 & -1/2 \\
\end{pmatrix}.
\]

The null space of \( A \) is spanned by \((-5/2, 1/2, 1)\). Thus \( \{(-5/2, 1/2, 1)\} \) is a basis for \( V \perp \).

b) The equation \(-\frac{5}{2}x + \frac{1}{2}y + z = 0\) is satisfied for \((x, y, z) = (1, 1, 2)\) and \((x, y, z) = (1, 3, 1)\).

c) We find \( b_{V \perp} \) first. We clear fractions by noting that \( V \perp \) is spanned by \((-5, 1, 2) = 2(-5/2, 1/2, 1)\). By the formula for projection onto a line, we have

\[
b_{V \perp} = \frac{1}{\begin{pmatrix}
1 \\
1 \\
\end{pmatrix} \cdot \begin{pmatrix}
-5 \\
1 \\
2 \\
\end{pmatrix}} \begin{pmatrix}
-5 \\
1 \\
2 \\
\end{pmatrix} = \frac{-2}{30} \begin{pmatrix}
-5 \\
1 \\
2 \\
\end{pmatrix} = -\frac{1}{15} \begin{pmatrix}
-5 \\
1 \\
2 \\
\end{pmatrix}.
\]

Then we compute

\[
b_v = b - b_{V \perp} = \begin{pmatrix}
1 \\
1 \\
\end{pmatrix} - \frac{1}{15} \begin{pmatrix}
-5 \\
1 \\
2 \\
\end{pmatrix} = \begin{pmatrix}
10/15 \\
16/15 \\
17/15 \\
\end{pmatrix} = \frac{1}{15} \begin{pmatrix}
10 \\
16 \\
17 \\
\end{pmatrix}.
\]
3. **Orthogonal projections, under the hood**

Consider the plane

\[ V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\} \]

in \( \mathbb{R}^4 \). We will find the orthogonal projection of \( b = (1, -1, -3, -5) \) onto \( V \), “by hand.” This is the vector \( b_V \in V \) satisfying \( b_V \perp b - b_V \in V^\perp \).

Since \( b_V \) is in \( V \), there exist scalars \( x_1, x_2 \) such that

\[ b_V = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}. \]

We will compute the orthogonal projection by solving for these scalars.

The vector \( b_V \perp \) is orthogonal to every vector in \( V \). In particular, it is orthogonal to both \( (1, 1, 1, 1) \) and \( (1, 2, 3, 4) \). We get two equations:

\[ (1, 1, 1, 1) \cdot b_V \perp = 0, \]
\[ (1, 2, 3, 4) \cdot b_V \perp = 0. \]

Expanding

\[ b_V \perp = b - b_V = \begin{pmatrix} 1 \\ -1 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \]

we can rewrite these two equations as

\[ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -3 \\ -5 \end{pmatrix}, \]

\[ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -3 \\ -5 \end{pmatrix}. \]

a) By computing the dot products, convert this into two linear equations in the two unknowns \( \tilde{x}_1 \) and \( \tilde{x}_2 \).

b) Solve for \( \tilde{x}_1 \) and \( \tilde{x}_2 \), and compute the orthogonal projection

\[ b_V = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}. \]

c) Confirm that the vector \( b_V \perp = b - b_V \) is orthogonal to \( V \) by checking that

\[ b_V \perp \cdot (1, 1, 1, 1) = 0 \quad \text{and} \quad b_V \perp \cdot (1, 2, 3, 4) = 0. \]
d) Write down a matrix $A$ whose column are the two vectors which span $V$, and compute $A^T A$, the matrix of column dot products. Compute the vector $A^T b$. Explain where the matrix equation $A^T A \tilde{x} = A^T b$ (the normal equation) appears in a) and b), and also where the product $b_V = A \tilde{x}$ appears.

**Solution.**

a) Using distributivity of dot products with respect to addition, these equations become

\[
4\tilde{x}_1 + 10\tilde{x}_2 = -8 \\
10\tilde{x}_1 + 30\tilde{x}_2 = -30.
\]

b) We solve these two equations by forming an augmented matrix and performing Gauss–Jordan elimination:

\[
\begin{pmatrix}
4 & 10 & -8 \\
10 & 30 & -30
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -2
\end{pmatrix}.
\]

Therefore $\tilde{x}_1 = 3$, $\tilde{x}_2 = -2$, and

\[
b_V = 3 \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} - 2 \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} = \begin{pmatrix}
1 \\
-1 \\
-3
\end{pmatrix}.
\]

(The fact that $b = b_V$ means that $b \in V$.)

c) $b_{V \perp} = b - b_V = 0$, so $b_{V \perp}$ is orthogonal to $V$.

d) $A^T A = \begin{pmatrix}
4 & 10 \\
10 & 30
\end{pmatrix}$ $A^T b = \begin{pmatrix}
-8 \\
-30
\end{pmatrix}$

The equation $A^T A \tilde{x} = A^T b$ is the same as the system of equations

\[
4\tilde{x}_1 + 10\tilde{x}_2 = -8 \\
10\tilde{x}_1 + 30\tilde{x}_2 = -30
\]

from a). The equation $b_V = A \tilde{x}$ is the same as the equation

\[
b_V = \tilde{x}_1 \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \tilde{x}_2 \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}.
\]
4. **Projection matrices for planes**

Consider the plane

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

in $\mathbb{R}^4$.

**a)** Compute the projection matrix $P_V$ for the subspace $V$. (Feel free to use a computer.)

**b)** Explain why the first two columns of $I_4 - P_V$ form a basis for $V^\perp$.

**c)** Use your answer to **b**) to describe the plane $V$ via two implicit equations:

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0, d_1 x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4 = 0 \right\}.$$

What are the coefficients $(c_1, c_2, c_3, c_4)$ and $(d_1, d_2, d_3, d_4)$, and why? Confirm that every vector in $V$ satisfies these equations by checking that both $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ do.

**Solution.**

**a)** The projection matrix is

$$P_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T.$$

We compute the inverse:

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} = \frac{1}{120-100} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 15 & -5 \\ -5 & 2 \end{pmatrix}.$$
Then

\[ P_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 15 & -5 \\ -5 & 2 \\ -5 & 3 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T \]

= \frac{1}{10} \begin{pmatrix} 10 & -3 \\ 5 & -1 \\ 0 & 1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}

= \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}.

b) We have

\[ P_{V^\perp} = I_4 - P_V = \frac{1}{10} \begin{pmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{pmatrix}. \]

Since \( V \) is a plane, we have \( \dim(V) = 2 \), so \( \dim(V^\perp) = 4 - \dim(V) = 2 \), and hence \( V^\perp \) is also a plane. The column space of a projection matrix is the subspace it projects on to, so \( \text{Col}(P_{V^\perp}) = V^\perp \). The first two columns of \( P_{V^\perp} \) are not collinear, so they form a basis for \( V^\perp \).

c) Scaling the first two columns of \( P_{V^\perp} \) by 10 gives another basis for \( V^\perp \), so

\[ V^\perp = \text{Span} \left\{ \begin{pmatrix} 3 \\ -4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \\ -2 \\ -1 \end{pmatrix} \right\} = \text{Col} \left( \begin{pmatrix} 3 & -4 \\ -4 & 7 \\ -1 & -2 \\ 2 & -1 \end{pmatrix} \right). \]

Now we use the relation \( V = (V^\perp)^\perp \) to conclude

\[ V = \text{Col} \left( \begin{pmatrix} 3 & -4 \\ -4 & 7 \\ -1 & -2 \\ 2 & -1 \end{pmatrix} \right)^\perp = \text{Nul} \left( \begin{pmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \end{pmatrix} \right) \]

= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{cases} 3x_1 - 4x_2 - x_3 + 2x_4 = 0 \\ -4x_1 + 7x_2 - 2x_3 - x_4 = 0 \end{cases}

Both equations are satisfied by the vectors \((1, 1, 1, 1)\) and \((1, 2, 3, 4)\)—this confirms that we have found correct equations.
5. Projection matrices for lines

For each line $L$, compute the projection matrix $P_L$.

a) $L = \text{Span}\{(1, 1)\}$  
b) $L = \text{Span}\{(1, 2, 3)\}$

c) $L = \{(x, y, z) \in \mathbb{R}^3 : 2x + y + z = 0\}^\perp$

Solution.

a) 

$$P_L = \frac{1}{(1) \cdot (1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

b) 

$$P_L = \frac{1}{(2) \cdot (1)} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

c) This line is equal to 

$$\text{Nul}\left( \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \right) = \text{Row}\left( \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \right) = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$  

Therefore, 

$$P_L = \frac{1}{(2) \cdot (1)} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$
6. Some mistakes to avoid

Here is a false “fact”:
Every projection matrix $P_V$ equals the identity matrix $I_n$.

Here is a false “proof”:

$$P_V = A(A^T A)^{-1} A^T = A A^{-1} (A^T)^{-1} A^T = (A A^{-1})(A^T)^{-1} A^T = I_n \cdot I_n = I_n.$$  

a) What is wrong would this proof?

b) In what case would this proof be correct?

Now consider the subspace $V = \text{Col}(A)$ for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) It would be incorrect to say that $P_V = A(A^T A)^{-1} A^T$ is the projection matrix onto $V$. Why?

**Hint:** Try computing $P_V$—what goes wrong?

d) Find a matrix $B$ so that $P_V = B(B^T B)^{-1} B^T$ is the projection matrix onto $V$—you do not need to compute $P_V$.

**Solution.**

a) The step $(A^T A)^{-1} = A^{-1} (A^T)^{-1}$ is incorrect—it only works when $A$ is square.

b) The proof is correct when $A$ is an invertible $n \times n$ matrix. In this case, $\text{Col}(A) = \mathbb{R}^n$ because $A$ has full row rank, so the projection matrix onto $\text{Col}(A)$ is indeed the identity matrix.

c) Since the columns of $A$ are not linearly independent, the matrix $A^T A$ is not invertible.

d) The first two columns of $A$ are its pivot columns. This means that, if we remove the third column of $A$, we get a new matrix $B$ with full column rank and $\text{Col}(B) = V$. We can use this matrix to compute $P_V = B(B^T B)^{-1} B^T$. 