Math 218D Problem Session: Week 6

Answer Key

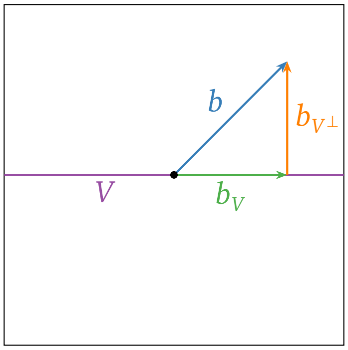
1. Projection onto a line

For each of the following pairs of vectors b and v,
(1) compute the orthogonal projection of b onto the line $V = \mathrm{Span}\{v\}$,
(2) draw V and the three vectors $b, b_V, b_{V^\perp}$, and
(3) compute the projection matrix $P_V = vv^T/(v^Tv)$.

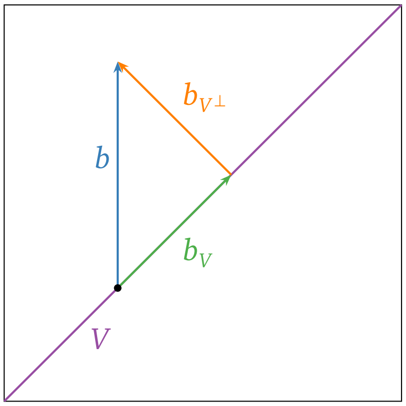
a) $b = \begin{pmatrix}1\\1\end{pmatrix},\ v = \begin{pmatrix}1\\0\end{pmatrix}$ **b)** $b = \begin{pmatrix}0\\2\end{pmatrix},\ v = \begin{pmatrix}1\\1\end{pmatrix}$ **c)** $b = \begin{pmatrix}1\\2\\3\end{pmatrix},\ v = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

Solution.

a) $b_V = \dfrac{b\cdot v}{v\cdot v}v = \begin{pmatrix}1\\0\end{pmatrix}$, $\quad b_{V^\perp} = b - b_V = \begin{pmatrix}0\\1\end{pmatrix}$, $\quad P_V = \dfrac{vv^T}{v^Tv} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$.



b) $b_V = \dfrac{b\cdot v}{v\cdot v}v = \begin{pmatrix}1\\1\end{pmatrix}$, $\quad b_{V^\perp} = b - b_V = \begin{pmatrix}-1\\1\end{pmatrix}$, $\quad P_V = \dfrac{vv^T}{v^Tv} = \dfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$.



c) $b_V = \dfrac{b\cdot v}{v\cdot v}v = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $\quad b_{V^\perp} = b - b_V = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\quad P_V = \dfrac{1}{3}\begin{pmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{pmatrix}$.

2. Planes and normal vectors

The subspace $V = \mathrm{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) Place the vectors $(1,1,2)$, $(1,3,1)$ into the rows of a 2×3 matrix A—this means that $\mathrm{Row}(A) = V$. Find a basis for $\mathrm{Nul}(A)$. Since

$$V^\perp = \mathrm{Row}(A)^\perp = \mathrm{Nul}(A),$$

you have found a basis vector $v = (a,b,c)$ for the line $V^\perp$.

In other words, you have found a basis for $V^\perp$ by solving the two orthogonality equations

$$(a,b,c)\cdot(1,1,2) = a+b+2c = 0,$$
$$(a,b,c)\cdot(1,3,1) = a+3b+c = 0.$$

b) Confirm that V is the plane $\{(x,y,z)\in\mathbf{R}^3 : ax+by+cz=0\}$, by showing that both $(1,1,2)$ and $(1,3,1)$ solve this equation. *The coefficients of a plane's equation make a normal vector for the plane.*

c) Find the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1,1,1)$ with respect to the plane V and the orthogonal line $V^\perp$.

Hint: It is easier to compute $b_{V^\perp}$, as it is the projection of b onto the line $V^\perp$ spanned by the vector $v = (a,b,c)$.

Solution.

a) We perform Gauss–Jordan elimination:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{pmatrix}.$$

The null space of A is spanned by $(-5/2, 1/2, 1)$. Thus $\{(-5/2,1/2,1)\}$ is a basis for $V^\perp$.

b) The equation $-\frac{5}{2}x + \frac{1}{2}y + z = 0$ is satisfied for $(x,y,z) = (1,1,2)$ and $(x,y,z) = (1,3,1)$.

c) We find $b_{V^\perp}$ first. We clear fractions by noting that $V^\perp$ is spanned by $(-5,1,2) = 2(-5/2,1/2,1)$. By the formula for projection onto a line, we have

$$b_{V^\perp} = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\1\\2\end{pmatrix}}{\begin{pmatrix}-5\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-5\\1\\2\end{pmatrix}}\begin{pmatrix}-5\\1\\2\end{pmatrix} = \frac{-2}{30}\begin{pmatrix}-5\\1\\2\end{pmatrix} = -\frac{1}{15}\begin{pmatrix}-5\\1\\2\end{pmatrix}.$$

Then we compute

$$b_V = b - b_{V^\perp} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \left(-\frac{1}{15}\begin{pmatrix}-5\\1\\2\end{pmatrix}\right) = \begin{pmatrix}10/15\\16/15\\17/15\end{pmatrix} = \frac{1}{15}\begin{pmatrix}10\\16\\17\end{pmatrix}.$$

3. Orthogonal projections, under the hood

Consider the plane

$$V = \operatorname{Span}\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right\}$$

in $\mathbf{R}^4$. We will find the orthogonal projection of $b = (1,-1,-3,-5)$ onto V, "by hand." This is the vector $b_V \in V$ satisfying $b_{V^\perp} = b - b_V \in V^\perp$.

Since b_V is in V, there exist scalars $\widehat{x}_1, \widehat{x}_2$ such that

$$b_V = \widehat{x}_1 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix}.$$

We will compute the orthogonal projection by solving for these scalars.

The vector $b_{V^\perp}$ is orthogonal to every vector in V. In particular, it is orthogonal to both $(1,1,1,1)$ and $(1,2,3,4)$. We get two equations:

$$(1,1,1,1)\cdot b_{V^\perp} = 0,$$
$$(1,2,3,4)\cdot b_{V^\perp} = 0.$$

Expanding

$$b_{V^\perp} = b - b_V = \begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix} - \widehat{x}_1 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \widehat{x}_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix},$$

we can rewrite these two equations as

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \cdot \left(\widehat{x}_1 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right) = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \cdot \left(\widehat{x}_1 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right) = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-3\\-5\end{pmatrix}.$$

a) By computing the dot products, convert this into two linear equations in the two unknowns $\widehat{x}_1$ and $\widehat{x}_2$.

b) Solve for $\widehat{x}_1$ and $\widehat{x}_2$, and compute the orthogonal projection

$$b_V = \widehat{x}_1 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \widehat{x}_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix}.$$

c) Confirm that the vector $b_{V^\perp} = b - b_V$ is orthogonal to V by checking that

$$b_{V^\perp}\cdot(1,1,1,1) = 0 \quad \text{and} \quad b_{V^\perp}\cdot(1,2,3,4) = 0.$$

d) Write down a matrix A whose column are the two vectors which span V, and compute $A^T A$, the *matrix of column dot products*. Compute the vector $A^T b$. Explain where the matrix equation $A^T A\widehat{x} = A^T b$ (the *normal equation*) appears in **a)** and **b)**, and also where the product $b_V = A\widehat{x}$ appears.

Solution.

a) Using distributivity of dot products with respect to addition, these equations become

$$\begin{aligned} 4\widehat{x}_1 + 10\widehat{x}_2 &= -8 \\ 10\widehat{x}_1 + 30\widehat{x}_2 &= -30. \end{aligned}$$

b) We solve these two equations by forming an augmented matrix and performing Gauss–Jordan elimination:

$$\left(\begin{array}{cc|c} 4 & 10 & -8 \\ 10 & 30 & -30 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right).$$

Therefore $\widehat{x}_1 = 3$, $\widehat{x}_2 = -2$, and

$$b_V = 3\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \\ -5 \end{pmatrix}.$$

(The fact that $b = b_V$ means that $b \in V$.)

c) $b_{V^\perp} = b - b_V = 0$, so $b_{V^\perp}$ is orthogonal to V.

d)

$$A^T A = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \qquad A^T b = \begin{pmatrix} -8 \\ -30 \end{pmatrix}$$

The equation $A^T A\widehat{x} = A^T b$ is the same as the system of equations

$$\begin{aligned} 4\widehat{x}_1 + 10\widehat{x}_2 &= -8 \\ 10\widehat{x}_1 + 30\widehat{x}_2 &= -30 \end{aligned}$$

from **a)**. The equation $b_V = A\widehat{x}$ is the same as the equation

$$b_V = \widehat{x}_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \widehat{x}_2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

4. Projection matrices for planes

Consider the plane

$$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

in $\mathbf{R}^4$.

a) Compute the projection matrix P_V for the subspace V. (Feel free to use a computer.)

b) Explain why the first two columns of $I_4 - P_V$ form a basis for $V^\perp$.

c) Use your answer to **b)** to describe the plane V via *two* implicit equations:

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbf{R}^4 \ : \ \begin{matrix} c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0 \\ d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0 \end{matrix} \right\}.$$

What are the coefficients (c_1, c_2, c_3, c_4) and (d_1, d_2, d_3, d_4), and why? Confirm that every vector in V satisfies these equations by checking that both $(1,1,1,1)$ and $(1,2,3,4)$ do.

Solution.

a) The projection matrix is

$$P_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T.$$

We compute the inverse:

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} = \frac{1}{120 - 100} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 15 & -5 \\ -5 & 2 \end{pmatrix}.$$

Then

$$P_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 15 & -5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T$$

$$= \frac{1}{10} \begin{pmatrix} 10 & -3 \\ 5 & -1 \\ 0 & 1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}.$$

b) We have

$$P_{V^\perp} = I_4 - P_V = \frac{1}{10} \begin{pmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{pmatrix}.$$

Since V is a plane, we have $\dim(V) = 2$, so $\dim(V^\perp) = 4 - \dim(V) = 2$, and hence $V^\perp$ is also a plane. The column space of a projection matrix is the subspace it projects on to, so $\mathrm{Col}(P_{V^\perp}) = V^\perp$. The first two columns of $P_{V^\perp}$ are not collinear, so they form a basis for $V^\perp$.

c) Scaling the first two columns of $P_{V^\perp}$ by 10 gives another basis for $V^\perp$, so

$$V^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 3 \\ -4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \\ -2 \\ -1 \end{pmatrix} \right\} = \mathrm{Col} \begin{pmatrix} 3 & -4 \\ -4 & 7 \\ -1 & -2 \\ 2 & -1 \end{pmatrix}.$$

Now we use the relation $V = (V^\perp)^\perp$ to conclude

$$V = \mathrm{Col} \begin{pmatrix} 3 & -4 \\ -4 & 7 \\ -1 & -2 \\ 2 & -1 \end{pmatrix}^\perp = \mathrm{Nul} \begin{pmatrix} 3 & -4 & -1 & 2 \\ -4 & 7 & -2 & -1 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbf{R}^4 \; : \; \begin{aligned} 3x_1 - 4x_2 - x_3 + 2x_4 &= 0 \\ -4x_1 + 7x_2 - 2x_3 - x_4 &= 0 \end{aligned} \right\}$$

Both equations are satisfied by the vectors $(1,1,1,1)$ and $(1,2,3,4)$—this confirms that we have found correct equations.

5. Projection matrices for lines

For each line L, compute the projection matrix P_L.

a) $L = \text{Span}\{(1,1)\}$ **b)** $L = \text{Span}\{(1,2,3)\}$

c) $L = \left\{(x,y,z) \in \mathbf{R}^3 : 2x + y + z = 0\right\}^{\perp}$

Solution.

a)
$$P_L = \frac{1}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

b)
$$P_L = \frac{1}{\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{14}\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{pmatrix}$$

c) This line is equal to

$$\text{Nul}\begin{pmatrix}2 & 1 & 1\end{pmatrix}^{\perp} = \text{Row}\begin{pmatrix}2 & 1 & 1\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}.$$

Therefore,

$$P_L = \frac{1}{\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}}\begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}4 & 2 & 2\\2 & 1 & 1\\2 & 1 & 1\end{pmatrix}.$$

6. Some mistakes to avoid

Here is a false "fact":

Every projection matrix P_V equals the identity matrix I_n.

Here is a false "proof":

$$P_V = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} A^T = (AA^{-1})((A^T)^{-1} A^T) = I_n \cdot I_n = I_n.$$

a) What is wrong would this proof?

b) In what case would this proof be correct?

Now consider the subspace $V = \mathrm{Col}(A)$ for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) It would be *incorrect* to say that $P_V = A(A^T A)^{-1} A^T$ is the projection matrix onto V. Why?

Hint: Try computing P_V—what goes wrong?

d) Find a matrix B so that $P_V = B(B^T B)^{-1} B^T$ is the projection matrix onto V—you do not need to compute P_V.

Solution.

a) The step $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$ is incorrect—it only works when A is *square*.

b) The proof is correct when A is an invertible $n \times n$ matrix. In this case, $\mathrm{Col}(A) = \mathbf{R}^n$ because A has full row rank, so the projection matrix onto $\mathrm{Col}(A)$ is indeed the identity matrix.

c) Since the columns of A are not linearly independent, the matrix $A^T A$ is not invertible.

d) The first two columns of A are its pivot columns. This means that, if we remove the third column of A, we get a new matrix B with full column rank and $\mathrm{Col}(B) = V$. We can use this matrix to compute $P_V = B(B^T B)^{-1} B^T$.