1. **Subspaces?**

Decide if each of the following sets of vectors is or is not a subspace. If so, express the subset as a null space or a column space of a matrix. If not, give a counterexample to one of the subspace axioms.

a) \( \{(x, y, z) \in \mathbb{R}^3 : x + y = 1 - z \} \)

b) \( \{(x, y) \in \mathbb{R}^2 : x - 2y = 0 \} \)

c) \( \{ v \in \mathbb{R}^3 : Av = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \} \) (A a 3 \( \times \) 3 matrix)

d) \( \{(x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \} \)

e) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \)

f) \( \{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + y^2 = 0 \} \)

**Solution.**

a) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

b) This is \( \text{Nul} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \).

c) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

d) We take transposes of both sides of the equation:

\[
\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}
\]

becomes

\[
\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Hence this subspace is the null space of the matrix

\[
\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}.
\]

e) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

f) This one is tricky. Note that \(x^2 + 2xy + y^2 = (x + y)^2\) is equal to zero if and only if \(x + y = 0\), so

\[
\{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + y^2 = 0 \} = \{(x, y) \in \mathbb{R}^2 : x + y = 0 \} = \text{Nul} \begin{pmatrix} 1 & 1 \end{pmatrix}.
\]
The four fundamental subspaces associated to a matrix $A$ are

$$\text{Nul}(A), \text{Col}(A), \text{Nul}(A^T), \text{Col}(A^T).$$

2. **The fundamental subspaces I**
Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$ 

**a)** Find a spanning set for each of the four fundamental subspaces of this matrix.

**b)** Draw each of the fundamental subspaces:

- **Nul($A$)**
- **Col($A$)**
- **Nul($A^T$)**
- **Col($A^T$)**

**c)** Compute $\dim(\text{Nul}(A)) + \dim(\text{Col}(A^T))$.

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**Solution.**

**a)** The subspaces $\text{Nul}(A)$ and $\text{Nul}(A^T)$ are points, while $\text{Col}(A)$ and $\text{Col}(A^T)$ are all of $\mathbb{R}^2$. Hence $\{\}$ is a spanning set for $\text{Nul}(A)$ and $\text{Nul}(A^T)$, and any pair of noncollinear vectors form a spanning set for $\text{Col}(A)$ and $\text{Col}(A^T)$.

**b)** We have to draw a point and a plane in each case:
c) \( \dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) = 2 \)
3. **The fundamental subspaces II**

Consider the matrix

\[ A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}. \]

**a)** Find a spanning set for each of the four fundamental subspaces of the matrix.

**b)** Draw each of the fundamental subspaces:

- **Nul(A)**
- **Col(A)**
- **Nul(A^T)**
- **Col(A^T)**

**c)** Compute \( \dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) \).

**d)** Describe the geometric relationship between \( \text{Nul}(A) \) and \( \text{Col}(A^T) \) and between \( \text{Col}(A) \) and \( \text{Nul}(A^T) \).

**Solution.**

**a)** One possible spanning set for each subspace is

\[ \text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \quad \text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \]

\[ \text{Nul}(A^T) = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad \text{Col}(A^T) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \]

Other answers are possible—you can scale each vector by a nonzero number, for instance.

**b)** We have to draw four lines:
c) \( \dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) = 2. \)

d) The lines \( \text{Nul}(A) \) and \( \text{Col}(A^T) \) are perpendicular. The lines \( \text{Col}(A) \) and \( \text{Nul}(A^T) \) are perpendicular.
4. The fundamental subspaces III

Consider the matrix

\[ A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}. \]

a) Is \( \text{Col}(A^T) \) a subspace of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)?

b) Is \( \text{Nul}(A) \) a subspace of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)?

c) Is \( \text{Col}(A) \) a subspace of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)?

d) Is \( \text{Nul}(A^T) \) a subspace of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)?

e) Two of the four subspaces are contained in \( \mathbb{R}^2 \). Draw these two subspaces, and describe the geometric relationship between them.

f) Two of the four subspaces are contained in \( \mathbb{R}^3 \). For this matrix, one is a line and the other is a plane. Determine which is which, and find bases for both of these subspaces.

g) Find an implicit equation \( a_1x + a_2y + a_3z = 0 \) for the plane in f).

h) What can you observe about the relationship between the answers to f) and g)? What does this mean geometrically?

Solution.

a) \( \text{Col}(A^T) \) is a subspace of \( \mathbb{R}^3 \).

b) \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^3 \).

c) \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^2 \).

d) \( \text{Nul}(A^T) \) is a subspace of \( \mathbb{R}^2 \).

e) We compute

\[
\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \quad \text{and} \quad \text{Nul}(A^T) = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.
\]

These lines are perpendicular:
f) The subspace $\text{Col}(A^T)$ is spanned by the vectors $(1, -1, 2)$ and $(-2, 2, -4)$, but these are scalar multiples of each other, so $\text{Col}(A^T)$ is a line; either vector forms a basis. The parametric vector form of the solution set of $Ax = 0$ is

$$
\begin{pmatrix} x \\
  y \\
  z
\end{pmatrix} = y \begin{pmatrix} 1 \\
  1 \\
  0
\end{pmatrix} + z \begin{pmatrix} -2 \\
  0 \\
  1
\end{pmatrix},
$$

so $(1, 1, 0)$ and $(-2, 0, 1)$ form a basis for $\text{Nul}(A)$.

g) The matrix equation $Ax = 0$ translates into the system of equations

\begin{align*}
x - y + 2z &= 0 \\
-2x + 2y - 4z &= 0.
\end{align*}

These equations are scalar multiples of each other, so the plane $\text{Nul}(A)$ is determined by either of these equations. For concreteness, we express $\text{Nul}(A)$ as the solution set of the first equation:

$$
x - y + 2z = 0.
$$

h) The coefficients of the equation above are $(1, -1, 2)$. This vector is a basis for $\text{Col}(A^T)$ (you may have gotten a scalar multiple of this vector in f)). This means that every vector in the plane is perpendicular to the vector $(1, -1, 2)$, i.e., that the plane has normal vector $(1, -1, 2)$. In other words, the null space is orthogonal to the row space. We will discuss the orthogonality of the fundamental subspaces in more detail next week.
5. Linear (in)dependence I
   
a) Are the vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) linearly independent? If not, write down a linear relation.

b) Are the vectors \( \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) linearly independent? If not, write down a linear relation.

c) What is the dimension of 
\[
\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} 
\]

   
d) Consider two linearly independent vectors \( u, v \in \mathbb{R}^n \). Show that the two vectors \( u + v, u - v \) are linearly independent.

   
e) Consider three vectors \( u, v, w \in \mathbb{R}^n \). Show that the three vectors \( u + v, u + 2v - w, v - w \) are linearly dependent.

f) Show that the vectors
\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]
are linearly dependent by writing down a linear relation among them.

Solution.

   a) Since neither vector is a scalar multiple of the other, the two vectors are linearly independent.

   b) Any three vectors in \( \mathbb{R}^2 \) must be linearly dependent. To find a linear relation, we solve the vector equation
\[
x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 0
\]
by solving the matrix equation \( Ax = 0 \), where
\[
A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}.
\]
The parametric vector form of the solution set of \( Ax = 0 \) is
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -5/2 \\ -1/2 \\ 1 \end{pmatrix}.
\]
Choosing \( x_3 = 2 \) gives the solution \( (x_1, x_2, x_3) = (-5, -1, 2) \), so
\[
-5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 0
\]
is a linear relation.
e) The dimension of the span is the same as the rank of the matrix

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \]

since the rank of a matrix equals the dimension of its column space. We put \( A \) in row echelon form:

\[
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

There are 3 pivots, so its rank is 3, and hence the span has dimension 3.

d) Consider any two scalars \( a, b \) such that

\[ a(u + v) + b(u - v) = 0. \]

We need to show that both of these scalars are in fact equal to 0. We rewrite the equation above as

\[ (a + b)u + (a - b)v = 0. \]

Since \( u \) and \( v \) are linearly independent, this implies that \( a + b = 0 \) and \( a - b = 0 \). It follows that \( a = b \) and \( a = -b \), which implies that \( a = b = 0 \). This means that \( u + v \) and \( u - v \) are linearly independent.

e) The vectors \( u + v, u + 2v - w, v - w \) are linearly dependent because

\[ (u + v) + (v - w) - (u + 2v - w) = 0. \]

f) We solve the vector equation

\[
x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]

by solving the matrix equation \( Ax = 0 \) for

\[
A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & -4 & 1 & 1 \end{pmatrix}.
\]

The parametric vector form of the solution set of \( Ax = 0 \) is

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \\ 1 \end{pmatrix}.
\]
Choosing $x_4 = 4$ gives the solution $(x_1, x_2, x_3, x_4) = (-1, 1, 1, 4)$, so

$$-\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a linear relation.
6. **Linear (in)dependence II**

For each of the following statements, find examples of a $2 \times 2$ matrix $A$ and vectors $u, v \in \mathbb{R}^2$ such that the statement holds. If it is impossible to do so, explain why.

   a) $u, v$ are linearly independent, but $Au, Av$ are linearly dependent.

   b) $A$ is invertible and $\{u, v\}$ are linearly independent, but $\{Au, Av\}$ is linearly dependent.

   c) $u, v$ are linearly dependent, but $Au, Av$ are linearly independent.

   d) $u, v$ are linearly dependent, but $Au, v$ are linearly independent.

**Solution.**

   a) Take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   b) This is impossible: if $\{u, v\}$ is linearly independent then $\{Au, Av\}$ is also linearly independent. To see this, suppose that $aAu + bAv = 0$. We can rewrite this as $A(au + bv) = 0$. Multiplying both sides by $A^{-1}$ gives $au + bv = A^{-1}0 = 0$. This implies $a = b = 0$ because $\{u, v\}$ is linearly independent. It follows that $\{Au, Av\}$ is also linearly independent.

   c) This is impossible: if $au + bv = 0$ is a linear relation between $u, v$ then

   $$0 = A(au + bv) = aAu + bAv$$

   is a linear relation between $Au, Av$.

   d) Take $u = v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. 