

Math 218D Problem Session: Week 3

September 16, 2022

1. Parametric forms

Consider the augmented matrix $(A | b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 \\ 0 & -3 & -2 & 1 \end{array} \right)$.

- a) Compute the RREF, and verify that the system of equations $Ax = b$ has one free variable. Which variable is it?
- b) Find the parametric form of the solution:

$$x_1 = (?)$$

$$x_2 = (?)$$

$$x_3 = (?)$$

where all the (?) only involve scalars and the free variable. What are two different solutions to the system of equations?

- c) Find the parametric vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix} + (\text{free variable}) \cdot \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix}$$

where all the (?) are scalars.

- d) This is the parametric vector form of a line. Find a point that this line passes through. What direction is the line pointing? Check your answer with this [demo](#).
- e) Let's modify the b -vector: $b = (0, 1, -1)$. How many solutions does this new system of equations have? Check your answer geometrically by moving the b vector in the demo linked above.
- f) Now find the parametric vector form of the homogeneous equation $Ax = 0$. How is this related to your answer in d)? Check your answer geometrically by moving the b vector in the demo linked above.
- g) Describe all the triples (b_1, b_2, b_3) which make

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & -2 & -1 & b_2 \\ 0 & -3 & -2 & b_3 \end{array} \right)$$

consistent. Your answer should involve a single linear equation in the variables b_1, b_2, b_3 .

- h) What shape (point, line, plane, ...) do you get if you add together the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

in every way possible? Does this shape contain the point $(0, 1, 1)$? How about $(0, 1, -1)$? or $(0, 0, 0)$? How does this relate to \mathbf{g} ? Check your answer with this [demo](#).

$$\text{Hint: } \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

2. Parallel lines

The solution set of

$$\begin{aligned}x + y + z &= 1 \\2x + 3y + z &= 0\end{aligned}$$

is a line L inside of \mathbf{R}^3 .

a) Describe the line L as the translate of a span, i.e., as

$$\text{Span} \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

What point does the line L pass through?

- b) Find two different points, P_1 and P_2 , on L . Verify that $P_2 - P_1$ is contained in $\text{Span}\{(v_1, v_2, v_3)\}$ from part a).
- c) Find a different system of two linear equations whose solution set is *parallel* to L , passing through the point $(1, 1, 1)$.

3. The geometry of spans

a) Is it possible to find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

Solve the system algebraically, then geometrically using this [demo](#).

b) Describe

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Is it a point, line, plane, or all of \mathbf{R}^3 ? How do you know? Check your answer with this [demo](#).

c) Find an implicit equation $ax + by + cz = d$ for the plane parametrized by

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

as we vary x_1 and x_2 .

Hint: Describe all the vectors $b = (x, y, z)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

consistent.

d) It is possible to find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}?$$

Explain why, *without* finding x_1 and x_2 . Then find x_1 and x_2 using this [demo](#).

e) Describe the span of the vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$. Is it a point, line, plane, or all of \mathbf{R}^3 ? How do you know? Check your answer with this [demo](#).

4. Column picture criterion

Consider the system of equations

$$\begin{aligned}3x_1 + x_2 - x_3 &= b_1 \\ -2x_1 + 3x_2 + 5x_3 &= b_2 \\ 5x_1 + 9x_2 + 7x_3 &= b_3.\end{aligned}$$

- a) When is the system consistent? Your answer should be a single linear equation in b_1, b_2, b_3 . [**Hint:** perform Gaussian elimination.]
- b) Explain the relationship between your answer in (a) and

$$\text{Span} \left\{ \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} \right\}.$$