1. **Parametric forms**

Consider the augmented matrix \( (A \mid b) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 \\ 0 & -3 & -2 & 1 \end{pmatrix} \).

a) Compute the RREF, and verify that the system of equations \( Ax = b \) has one free variable. Which variable is it?

b) Find the parametric form of the solution:
\[
\begin{align*}
  x_1 &= (?) \\
  x_2 &= (?) \\
  x_3 &= (?)
\end{align*}
\]
where all the (?) only involve scalars and the free variable. What are two different solutions to the system of equations?

c) Find the parametric vector form:
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix} + \text{(free variable)} \cdot \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix}
\]
where all the (?) are scalars.

d) This is the parametric vector form of a line. Find a point that this line passes through. What direction is the line pointing? Check your answer with this demo.

e) Let's modify the \( b \)-vector: \( b = (0, 1, -1) \). How many solutions does this new system of equations have? Check your answer geometrically by moving the \( b \) vector in the demo linked above.

f) Now find the parametric vector form of the homogeneous equation \( Ax = 0 \). How is this related to your answer in d)? Check your answer geometrically by moving the \( b \) vector in the demo linked above.

g) Describe all the triples \( (b_1, b_2, b_3) \) which make
\[
\begin{pmatrix} 1 & 1 & 1 & b_1 \\ 1 & -2 & -1 & b_2 \\ 0 & -3 & -2 & b_3 \end{pmatrix}
\]
consistent. Your answer should involve a single linear equation in the variables \( b_1, b_2, b_3 \).

h) What shape (point, line, plane, ...) do you get if you add together the vectors
\[
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}
\]
in every way possible? Does this shape contain the point \((0, 1, 1)\)? How about \((0, 1, -1)\) or \((0, 0, 0)\)? How does this relate to \(\text{g)}\)? Check your answer with this demo.

**Hint:**

\[
\begin{pmatrix}
  1 & 1 & 1 \\
  1 & -2 & -1 \\
  0 & -3 & -2
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= x_1
\begin{pmatrix}
  1 \\
  1 \\
  0
\end{pmatrix}
+ x_2
\begin{pmatrix}
  1 \\
  -2 \\
  -3
\end{pmatrix}
+ x_3
\begin{pmatrix}
  1 \\
  -1 \\
  2
\end{pmatrix}.
\]

**Solution.**

**a)** The RREF is

\[
\begin{pmatrix}
  1 & 0 & 1/3 \\
  0 & 1 & 2/3 \\
  0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  1/3 \\
  -1/3 \\
  0
\end{pmatrix}
\]

**b)** The parametric form of the solution is:

\[
\begin{align*}
    x_1 &= \frac{1}{3} - \frac{1}{2} x_3 \\
    x_2 &= -\frac{1}{3} - \frac{2}{3} x_3 \\
    x_3 &= x_3
\end{align*}
\]

Setting \(x_3 = 0\) gives one solution: \((x_1, x_2, x_3) = (1/3, -1/3, 0)\). Setting \(x_3 = 1\) gives another solution \((x_1, x_2, x_3) = (0, -1, 1)\).

**c)** The parametric vector form is:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= \frac{1}{3}
\begin{pmatrix}
  1 \\
  -1 \\
  0
\end{pmatrix}
+ x_3
\begin{pmatrix}
  -1/3 \\
  -2/3 \\
  1
\end{pmatrix}.
\]

**d)** The line passes through the point \((1/3, -1/3, 0)\), and goes in the direction of the vector \((-1/3, -2/3, 1)\).

**e)** This system of equations has no solutions.

**f)** The parametric vector form of this system is

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= x_3
\begin{pmatrix}
  -1/3 \\
  -2/3 \\
  1
\end{pmatrix}.
\]

The solution to the homogeneous equation is a line, parallel to the line from part \(\text{d)}\), passing through the origin.

**g)** A vector \(b = (b_1, b_2, b_3)\) makes \(Ax = b\) consistent precisely when \(-b_1 + b_2 - b_3 = 0\).

**h)** The span of these vectors is the same as the set of vectors making \(Ax = b\) consistent. By \(g)\), this is the same as the vectors which satisfying a single linear equation. The set of vectors satisfying a single linear equation is a plane.
2. **Parallel lines**

The solution set of

\[
\begin{align*}
    x + y + z &= 1 \\
    2x + 3y + z &= 0
\end{align*}
\]

is a line \( L \) inside of \( \mathbb{R}^3 \).

(a) Describe the line \( L \) as the translate of a span, i.e., as

\[
\text{Span} \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.
\]

What point does the line \( L \) pass through?

(b) Find two different points, \( P_1 \) and \( P_2 \), on \( L \). Verify that \( P_2 - P_1 \) is contained in \( \text{Span}\{ (v_1, v_2, v_3) \} \) from part (a).

(c) Find a different system of two linear equations whose solution set is parallel to \( L \), passing through the point \((1,1,1)\).

**Solution.**

(a) \[
\text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}.
\]

(b) One possible answer is

\[
P_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad P_2 - P_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.
\]

(c) To get a parallel line, you need the same matrix \( A \) but a different \( b \) vector. You can find the correct \( b \) vector by multiplying \( A \) times \( x = (1,1,1) \): \( A(1,1,1) = (3,6) \). In other words, the solution set of

\[
\begin{align*}
    x + y + z &= 3 \\
    2x + 3y + z &= 6
\end{align*}
\]

is parallel to \( L \) and passes through \((1,1,1)\).
3. The geometry of spans

a) Is it possible to find scalars $x_1, x_2$ so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$ 

Solve the system algebraically, then geometrically using this demo.

b) Describe

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Is it a point, line, plane, or all of $\mathbb{R}^3$? How do you know? Check your answer with this demo.

c) Find an implicit equation $ax + by + cz = d$ for the plane parametrized by

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$
as we vary $x_1$ and $x_2$.

**Hint:** Describe all the vectors $b = (x, y, z)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
consistent.

d) It is possible to find scalars $x_1, x_2$ so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}.$$ 

Explain why, without finding $x_1$ and $x_2$. Then find $x_1$ and $x_2$ using this demo.

e) Describe the span of the vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$. Is it a point, line, plane, or all of $\mathbb{R}^3$? How do you know? Check your answer with this demo.
Solution.

a) No, it is not possible. You can confirm this by Gaussian elimination:

\[
\begin{pmatrix}
1 & 1 & 1 \\
-1 & -1 & 1 \\
5 & 1 & 0
\end{pmatrix}
\xrightarrow{\text{REF}}
\begin{pmatrix}
1 & 1 & 1 \\
0 & -4 & -5 \\
0 & 0 & 2
\end{pmatrix}.
\]

Alternately, you could observe that the first two components of \((1, -1, 5)\) and \((1, -1, 1)\) add up to 0, while the first two components of \((1, 1, 0)\) do not.

b) It is all of \(\mathbb{R}^3\), since \(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\) is not contained in the plane \(\text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}\) by part a).

c) Again we use Gaussian elimination:

\[
\begin{pmatrix}
1 & 1 & x \\
-1 & -1 & y \\
5 & 1 & z
\end{pmatrix}
\xrightarrow{\text{REF}}
\begin{pmatrix}
1 & 1 & x \\
0 & -4 & z - 5x \\
0 & 0 & x + y
\end{pmatrix}.
\]

This consistent exactly when \(x + y = 0\), so a vector \(b = (x, y, z)\) makes

\[
\begin{pmatrix}
1 & 1 \\
-1 & -1 \\
5 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

consistent exactly when \(x + y = 0\). In other words, the plane parametrized by

\[x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\]

has equation \(x + y = 0\).

d) Yes, it is possible to find scalars \(x_1, x_2\) so that

\[x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},\]

since \((4, -4, 0)\) solves the equation \(x + y = 0\) found in c).

e) The vectors \((1, -1, 5)\) and \((1, -1, 1)\) are not parallel, so they span a plane. The third vector \((4, -4, 0)\) is contained in that plane by d), so adding it to the list of vectors does not enlarge the span.
4. Column picture criterion
Consider the system of equations
\[
3x_1 + x_2 - x_3 = b_1 \\
-2x_1 + 3x_2 + 5x_3 = b_2 \\
5x_1 + 9x_2 + 7x_3 = b_3.
\]
a) When is the system consistent? You answer should be a single linear equation in \( b_1, b_2, b_3 \). \textbf{[Hint: perform Gaussian elimination.]}

b) Explain the relationship between your answer in (a) and \( \text{Span} \left\{ \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ -5 \\ 7 \end{pmatrix} \right\} \).

\textbf{Solution.}

a) We perform Gaussian elimination:
\[
\begin{pmatrix}
3 & 1 & -1 & \vert & b_1 \\
-2 & 3 & 5 & \vert & b_2 \\
5 & 9 & 7 & \vert & b_3
\end{pmatrix}
\xrightarrow{\text{REF}}
\begin{pmatrix}
3 & 1 & -1 & \vert & b_1 \\
0 & 13/3 & 13/3 & \vert & b_2 + 2/3b_1 \\
0 & 0 & 0 & \vert & b_3 - 2b_2 - 3b_1
\end{pmatrix}.
\]
The system is consistent exactly when \( b_3 - 2b_2 - 3b_1 = 0 \), or equivalently, when \( -3b_1 - 2b_2 + b_3 = 0 \).

b) According to the column picture criterion of consistency, the system is consistent if and only if
\[
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{Span} \left\{ \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ -5 \\ 7 \end{pmatrix} \right\}.
\]
Hence the equation \( -3b_1 - 2b_2 + b_3 = 0 \) from a) is an implicit form of the plane given in parametric form by the above span.