Math 218D Problem Session: Week 3

Answer Key

1. Parametric forms

Consider the augmented matrix $\left(A \mid b\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 \\ 0 & -3 & -2 & 1 \end{array}\right)$.

a) Compute the RREF, and verify that the system of equations $Ax = b$ has one free variable. Which variable is it?

b) Find the parametric form of the solution:

$$\begin{aligned} x_1 &= (?) \\ x_2 &= (?) \\ x_3 &= (?) \end{aligned}$$

where all the (?) only involve scalars and the free variable. What are two different solutions to the system of equations?

c) Find the parametric vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix} + (\text{free variable}) \cdot \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix}$$

where all the (?) are scalars.

d) This is the parametric vector form of a line. Find a point that this line passes through. What direction is the line pointing? Check your answer with this demo.

e) Let's modify the b-vector: $b = (0, 1, -1)$. How many solutions does this new system of equations have? Check your answer geometrically by moving the b vector in the demo linked above.

f) Now find the parametric vector form of the homogeneous equation $Ax = 0$. How is this related to your answer in **d)**? Check your answer geometrically by moving the b vector in the demo linked above.

g) Describe all the triples (b_1, b_2, b_3) which make

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & -2 & -1 & b_2 \\ 0 & -3 & -2 & b_3 \end{array}\right)$$

consistent. Your answer should involve a single linear equation in the variables b_1, b_2, b_3.

h) What shape (point, line, plane, ...) do you get if you add together the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

in every way possible? Does this shape contain the point $(0,1,1)$? How about $(0,1,-1)$? or $(0,0,0)$? How does this relate to **g)**? Check your answer with this demo.

$$\textbf{Hint: } \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Solution.

a) The RREF is $\left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) The parametric form of the solution is:

$$\begin{aligned} x_1 &= \tfrac{1}{3} - \tfrac{1}{3}x_3 \\ x_2 &= -\tfrac{1}{3} - \tfrac{2}{3}x_3 \\ x_3 &= x_3 \end{aligned} .$$

Setting $x_3 = 0$ gives one solution: $(x_1, x_2, x_3) = (1/3, -1/3, 0)$. Setting $x_3 = 1$ gives another solution $(x_1, x_2, x_3) = (0, -1, 1)$.

c) The parametric vector form is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix}.$$

d) The line passes through the point $(1/3, -1/3, 0)$, and goes in the direction of the vector $(-1/3, -2/3, 1)$.

e) This system of equations has no solutions.

f) The parametric vector form of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix}.$$

The solution to the homogeneous equation is a line, parallel to the line from part d), passing through the origin.

g) A vector $b = (b_1, b_2, b_3)$ makes $Ax = b$ consistent precisely when $-b_1 + b_2 - b_3 = 0$.

h) The span of these vectors is the same as the set of vectors making $Ax = b$ consistent. By g), this is the same as the vectors which satisfying a single linear equation. The set of vectors satisfying a single linear equation is a plane.

2. Parallel lines

The solution set of

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + z &= 0 \end{aligned}$$

is a line L inside of $\mathbf{R}^3$.

a) Describe the line L as the translate of a span, i.e., as

$$\operatorname{Span}\left\{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right\} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

What point does the line L pass through?

b) Find two different points, P_1 and P_2, on L. Verify that $P_2 - P_1$ is contained in $\operatorname{Span}\{(v_1, v_2, v_3)\}$ from part **a)**.

c) Find a different system of two linear equations whose solution set is *parallel* to L, passing through the point $(1, 1, 1)$.

Solution.

a)

$$\operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}.$$

b) One possible answer is

$$P_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \qquad P_2 - P_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

c) To get a parallel line, you need the same matrix A but a different b vector. You can find the correct b vector by multiplying A times $x = (1, 1, 1)$: $A(1, 1, 1) = (3, 6)$. In other words, the solution set of

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 6 \end{aligned}$$

is parallel to L and passes through $(1, 1, 1)$.

3. The geometry of spans

a) Is it possible to find scalars x_1, x_2 so that

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}?$$

Solve the system algebraically, then geometrically using this demo.

b) Describe

$$\text{Span}\left\{\begin{pmatrix}1\\-1\\5\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}.$$

Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.

c) Find an implicit equation $ax+by+cz=d$ for the plane parametrized by

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

as we vary x_1 and x_2.

Hint: Describe all the vectors $b=(x,y,z)$ which make

$$\begin{pmatrix}1&1\\-1&-1\\5&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

consistent.

d) It is possible to find scalars x_1, x_2 so that

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}4\\-4\\0\end{pmatrix}?$$

Explain why, *without* finding x_1 and x_2. Then find x_1 and x_2 using this demo.

e) Describe the span of the vectors $\begin{pmatrix}1\\-1\\5\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}4\\-4\\0\end{pmatrix}$. Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.

Solution.

a) No, it is not possible. You can confirm this by Gaussian elimination:

$$\left(\begin{array}{rr|r} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 5 & 1 & 0 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & -4 & -5 \\ 0 & 0 & 2 \end{array}\right).$$

Alternately, you could observe that the first two components of $(1, -1, 5)$ and $(1, -1, 1)$ add up to 0, while the first two components of $(1, 1, 0)$ do not.

b) It is all of $\mathbf{R}^3$, since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is not contained in the plane $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ by part **a)**.

c) Again we use Gaussian elimination:

$$\left(\begin{array}{rr|r} 1 & 1 & x \\ -1 & -1 & y \\ 5 & 1 & z \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{rr|r} 1 & 1 & x \\ 0 & -4 & z - 5x \\ 0 & 0 & x + y \end{array}\right).$$

This consistent exactly when $x + y = 0$, so a vector $b = (x, y, z)$ makes

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

consistent exactly when $x + y = 0$. In other words, the plane parametrized by

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

has equation $x + y = 0$.

d) Yes, it is possible to find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},$$

since $(4, -4, 0)$ solves the equation $x + y = 0$ found in **c)**.

e) The vectors $(1, -1, 5)$ and $(1, -1, 1)$ are not parallel, so they span a plane. The third vector $(4, -4, 0)$ is contained in that plane by **d)**, so adding it to the list of vectors does not enlarge the span.

4. Column picture criterion

Consider the system of equations

$$\begin{aligned} 3x_1 + x_2 - x_3 &= b_1 \\ -2x_1 + 3x_2 + 5x_3 &= b_2 \\ 5x_1 + 9x_2 + 7x_3 &= b_3. \end{aligned}$$

a) When is the system consistent? You answer should be a single linear equation in b_1, b_2, b_3. [**Hint:** perform Gaussian elimination.]

b) Explain the relationship between your answer in (a) and

$$\text{Span}\left\{ \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} \right\}.$$

Solution.

a) We perform Gaussian elimination:

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & b_1 \\ -2 & 3 & 5 & b_2 \\ 5 & 9 & 7 & b_3 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 3 & 1 & -1 & b_1 \\ 0 & \frac{11}{3} & \frac{13}{3} & b_2 + \frac{2}{3}b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 - 3b_1 \end{array}\right).$$

The system is consistent exactly when $b_3 - 2b_2 - 3b_1$, or equivalently, when $-3b_1 - 2b_2 + b_3 = 0$.

b) According to the column picture criterion of consistency, the system is consistent if and only if

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} \right\}.$$

Hence the equation $-3b_1 - 2b_2 + b_3 = 0$ from **a)** is an *implicit* form of the plane given in *parametric* form by the above span.